Mathematics 2130
Project 1
The Bisection Method

A fundamental problem in mathematics is **root-finding**. Given a function $f(x)$, we wish to determine all values of x (or, frequently, all *real* values of x) which satisfy the equation

$$f(x) = 0.$$

If $f(x)$ is not too complicated, we might be able to compute exact values of the roots. In general, however, this is not practical; in such cases, we can often turn to a **numerical** technique, implemented via a computer code, to solve for x.

There are many different root-finding algorithms in the literature. In this project, we will concentrate on one of the simplest such techniques, called the **bisection method**. Here we begin with a continuous function $f(x)$ and an interval $I_0 = [a, b]$ for which $f(a)$ and $f(b)$ have different signs. Thus $f(x)$ must have at least one real root on I_0. **(WHY?)** We then compute the midpoint

$$m = \frac{a+b}{2}$$

of I_0. This is our first approximation of the root. If $f(m) = 0$ then we have found the root exactly... but we aren't usually that lucky! Instead, we compute $f(m)$. If it has the same sign as $f(a)$, then a root of $f(x)$ must lie on the interval $I_1 = [m, b]$. On the other hand, if $f(m)$ and $f(b)$ have the same sign, then the root must lie on the interval $I_1 = [a, m]$. Once we have identified the appropriate interval I_1, we then compute its midpoint to obtain a new approximation of the root, and repeat this process a sufficient number of times.

Methodology

We will begin by studying a simple function to which we can apply the bisection method:

$$f(x) = 3x^2 - 4x - 1.$$

Write a code to implement the bisection method for $f(x)$. You should be able to easily consider different choices of the endpoints a and b. As output, provide the approximation m to a root of $f(x)$ at each step of the algorithm. For this particular function $f(x)$, you should be able to determine two different roots, depending on your choice of I_0.

> By 11:59pm on Thursday, January 20th you must submit three sets of output from your code. Two should lead to the identification of a different root. The other should demonstrate how your code behaves if $f(a)$ and $f(b)$ have the same sign. They can be submitted to me as text files via e-mail.

Fortunately, $f(x) = 3x^2 - 4x - 1$ is a sufficiently simple function that we can compute its roots exactly using the **quadratic formula**. Give a full derivation of this formula, and use it to confirm your numerical results. (Be sure to include this in your finished report!)

> By 11:59pm on Thursday, January 27th you must submit a clear, properly-typeset derivation of the quadratic formula. Both the .tex file and the .pdf file should be submitted to me via e-mail.

Next, prepare a technical report about the bisection method, which should include (but need not be limited to) the following:

- A detailed explanation of how the bisection method works. Mathematically, why is the bisection method guaranteed to find a root?
- A discussion of the benefits and disadvantages of the bisection method. Modify your code to investigate a number of functions, including $f(x) = 3x^2 - 4x - 1$, that illustrate interesting properties of the bisection method. For instance, the **rate of convergence** of a numerical scheme — the speed at which the algorithm produces a result — is often of concern. Is there a way to estimate the number of steps required by the bisection method to locate a root?
- An explanation of appropriate **stopping criteria** which will ensure that your code outputs a root to good accuracy, and terminates if it is unable to find a root suitably quickly.
- The results of your computer code, given in clear, easily-interpreted tables. If a particular implementation of your code requires many steps to locate a root, give careful thought as to how best to present your data.
- Any other issues or topics related to the bisection method that are relevant to your discussion. For example, the bisection method is part of a class of **bracketing** methods that also includes the **method of false position**; you might discuss the differences between these two techniques and their respective advantages and disadvantages.

Remember that all key mathematical ideas and results should be carefully and formally explained; if these ideas and results are not of your own invention, citations must be provided.