Mathematics 2130
Project 1
Euler's Method

A **differential equation** is an equation involving an unknown function y and one or more of its derivatives. If y is a function of a single variable — say, $y = y(x)$ — then we call the differential equation an **ordinary** differential equation (ODE). Furthermore, if the ordinary differential equation involves only the first derivative of y, it is called a **first-order** ordinary differential equation, and can be written in the form

$$\frac{dy}{dx} = f(x, y). \tag{1}$$

Some first-order ODEs can be solved very easily using basic calculus. Most notably, an ODE is **separable** if $f(x, y) = F(x)G(y)$, because then we can solve the equation by grouping the terms involving y on one side, the terms involving x on the other side, and then integrating both sides with respect to the appropriate variable:

$$\frac{dy}{dx} = F(x)G(y) \quad \Longrightarrow \quad \int \frac{1}{G(y)}\,dy = \int F(x)\,dx.$$

However, the integration could be very difficult (or even impossible). Moreover, many ODEs are not separable. As a result, we often turn to a **numerical** technique, implemented via a computer code, to solve the equation.

In the literature, there are many different methods for solving ODEs numerically. In this project, we will concentrate on one of the simplest such techniques, called **Euler's method**. Here we begin with a known value of $y(x)$, say $y(a) = \alpha$, called an **initial condition**. Our goal is to determine some other value of the function, say $y(b)$. We divide the interval $[a, b]$ into n subintervals $[x_i, x_{i+1}]$ where $x_0 = a$ and $x_n = b$. We let each subinterval be the same length h, called the **stepsize**. Then Euler's method allows us to approximate the value of the unknown function y at each of these points x_i. If we call these approximations z_i, then Euler's method states that

$$z_i = z_{i-1} + hf(x_{i-1}, z_{i-1}), \tag{2}$$

where $z_0 = y(x_0) = y(a) = \alpha$.

Methodology

First, write a code to implement Euler's method to solve the ODE

$$\frac{dy}{dx} = \frac{y(2-y)}{x+3},$$

that is, Equation (1) where

$$f(x, y) = \frac{y(2-y)}{x+3}.$$

Your code should accept the endpoints a and b as user input (either by prompt or as command-line arguments), as well as values for α and h. However, you may hard-code $f(x, y)$ into your program. Use your program to approximate $y(8)$, given the initial condition $y(0) = 6$. (In other words, let $a = 0$, $\alpha = 6$ and $b = 8$.) Try different values of h, such as $h = 1$, $h = 0.5$, $h = 0.1$ and so on.

As output, your code should provide the approximation z_i to the solution $y(x_i)$ at each step of the algorithm.

Your code is due by 12:00 NOON on Friday, September 15th.

Fortunately, this ODE is a separable equation that we can solve via the method described above. Explain in detail how to solve for the solution $y(x)$ of the ODE. Use the initial condition $y(0) = 6$ to solve for the constant of integration. Then determine the true value of $y(8)$. You shouldn't just write mathematics: be sure to explain key steps as you go, always writing in complete sentences.

A clear, properly-typeset solution of this equation is due by
12:00 NOON on Friday, September 22nd.

Next, prepare a report following the *Suggested Report Format* which discusses Euler's method. Your report should include (but need not be limited to) the following:

- A detailed derivation of Euler's method, explaining clearly how Equation (2) is found.
- Your numerical and analytical solutions of Equation (1) with

  $$f(x, y) = \frac{y(2-y)}{x+3}, \quad y(0) = 6$$

  for $y(8)$. (You'll probably want to include a version of your September 22nd submission in your finished paper.) Give the results of your computer code in clear, easily-interpreted tables. If you use a very small stepsize, give careful thought as to how to best present your data.
- A discussion of the benefits and disadvantages of Euler's method. For example, when solving Equation (1) with

  $$f(x, y) = \frac{y(2-y)}{x+3}, \quad y(0) = 6$$

  for $y(8)$, how do your numerical and analytical solutions compare? For each stepsize you used, compute the **absolute error** in your approximation by finding the absolute value of the difference between your approximation and the true value of $y(8)$ you found analytically. Is there a relationship between the error and h? Are there any other informative ways to quantify the error in the approximation? Provide illustration by modifying your code to consider other functions $f(x, y)$ of your own choosing. Your paper should include a discussion of <u>at least</u> one other ODE which is <u>meaningfully</u> different from the one given in these instructions.
- Any other issues or topics related to Euler's method that are relevant to your discussion. For example, there are several variations on Euler's method that can be found in the literature; you might explain and implement one of these techniques, and illustrate its advantages and disadvantages with respect to the standard version of Euler's method.

In preparing your report, you should make reference to appropriate sources of information. Your research should <u>not</u> be solely confined to Internet websites. You <u>must</u> include at least one reference to a non-electronic source related to the substance of this project; this could be a journal article or a specialised textbook, but it should not be a general textbook (such as a first-year calculus text).

Remember that all key mathematical ideas and results should be carefully and formally explained; if these ideas and results are not of your own invention, citations must be provided.