MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

WORKSHEET | **Mathematics 2051** | FALL 2007

For practise only. Not to be submitted.

1. (a) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation for which

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}3\\-1\\1\end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}4\\5\\-2\end{bmatrix}.$$

If $\underline{x} = \begin{bmatrix}a\\b\end{bmatrix}$ is any vector in $\mathbb{R}^2$, determine $T(\underline{x})$.

(b) Let $T : \mathbb{R}^2 \to P_2$ be a linear transformation for which

$$T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = 1 + 2x + 3x^3 \quad \text{and} \quad T\left(\begin{bmatrix}-2\\3\end{bmatrix}\right) = -2 - 4x + 7x^2.$$

If $\underline{x} = \begin{bmatrix}a\\b\end{bmatrix}$ is any vector in $\mathbb{R}^2$, determine $T(\underline{x})$.

2. (a) For the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix}a\\b\\c\end{bmatrix}\right) = \begin{bmatrix}a - 2b + c\\a + 5b - c\end{bmatrix}$$

find a matrix A such that $T = T_A$ (left multiplication by A).

(b) If $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation such that

$$T\left(\begin{bmatrix}1\\-3\end{bmatrix}\right) = \begin{bmatrix}-1\\0\\-1\end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix}2\\-5\end{bmatrix}\right) = \begin{bmatrix}-1\\-1\\0\end{bmatrix},$$

find a matrix A such that $T = T_A$.

3. Let the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ be defined by

$$T\left(\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\right) = \begin{bmatrix}x_1 - x_4\\x_2 - x_3\\x_1 + x_2 - x_3 - x_4\end{bmatrix}.$$

Find a matrix A such that $T = T_A$. Find a basis for $\ker(T)$ and $\operatorname{im}(T)$, and determine the nullity and the rank of T.

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4. Show that if a linear transformation $T : V \to W$ between vector spaces V and W is one-to-one and $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_n\}$ is a linearly independent set in V then $\{T(\underset{\sim}{x}_1), T(\underset{\sim}{x}_2), \ldots, T(\underset{\sim}{x}_n)\}$ is a linearly independent set in W.

5. Verify that each of the following linear transformations T is an isomorphism.

   (a) $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a+b \\ b+c \\ c+a \end{bmatrix}$$

   (b) $T : M_{22} \to P_3$ given by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b) + dx + cx^2 + (a-b)x^3$$

6. If $S : P_2 \to P_2$ and $T : P_2 \to P_2$ are linear transformations such that

$$S(p(x)) = p(0) + p(1)x + p(2)x^2 \quad \text{and} \quad T(a + bx + cx^2) = b + cx + ax^2,$$

   determine $ST(p(x))$ and $TS(p(x))$ for some vector $p(x)$ in P_2.

7. Show that if V, W and U are vector spaces, and $T : V \to W$ and $S : W \to U$ are one-to-one linear transformations, then ST is also one-to-one.

8. Determine whether the linear transformation $T : M_{22} \to M_{22}$ given by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a-c & b-d \\ 2a-c & 2b-d \end{bmatrix}$$

   has an inverse. If so, find $T^{-1}(A)$ for some general matrix $A = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ in M_{22}.