# MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

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WORKSHEET **Mathematics 2051** FALL 2007

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**SOLUTIONS**

1. (a) We have

$$\underline{x} = \begin{bmatrix} a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T(\underline{x}) = aT\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + bT\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= a \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 4 \\ 5 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3a + 4b \\ -a + 5b \\ a - 2b \end{bmatrix}.$$

(b) First we need to see how to write $\underline{x}$ as a linear combination of the given vectors. We let

$$\begin{bmatrix} a \\ b \end{bmatrix} = k \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \ell \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} k - 2\ell \\ 3\ell - k \end{bmatrix}$$

so $a = k - 2\ell$ and $b = 3\ell - k$. Adding these gives $\ell = a + b$ and so $k = a + 2(a+b) = 3a + 2b$. Hence

$$\underline{x} = \begin{bmatrix} a \\ b \end{bmatrix} = (3a + 2b) \begin{bmatrix} 1 \\ -1 \end{bmatrix} + (a + b) \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$T(\underline{x}) = (3a + 2b)T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) + (a + b)T\left(\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right)$$

$$= (3a + 2b)(1 + 2x + 3x^3) + (a + b)(-2 - 4x + 7x^2)$$

$$= a + 2ax + 7(a + b)x^2 + 3(3a + 2b)x^3.$$

2. (a) Observe that

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 5 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

so $A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 5 & -1 \end{bmatrix}$.

(b) We need to express the standard basis vectors as linear combinations of the given vectors. First we let

$$\begin{bmatrix}1\\0\end{bmatrix} = k\begin{bmatrix}1\\-3\end{bmatrix} + \ell\begin{bmatrix}2\\-5\end{bmatrix} = \begin{bmatrix}k+2\ell\\-3k-5\ell\end{bmatrix}$$

so $k + 2\ell = 1$ and $-3k - 5\ell = 0$. The second equation tells us that $\ell = -\frac{3}{5}k$, yielding $-\frac{1}{5}k = 1$. Thus $k = -5$ and $\ell = 3$. Then

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = -5T\left(\begin{bmatrix}1\\-3\end{bmatrix}\right) + 3T\left(\begin{bmatrix}2\\-5\end{bmatrix}\right) = -5\begin{bmatrix}-1\\0\\-1\end{bmatrix} + 3\begin{bmatrix}-1\\-1\\0\end{bmatrix} = \begin{bmatrix}2\\-3\\5\end{bmatrix}.$$

Next we let

$$\begin{bmatrix}0\\1\end{bmatrix} = k\begin{bmatrix}1\\-3\end{bmatrix} + \ell\begin{bmatrix}2\\-5\end{bmatrix} = \begin{bmatrix}k+2\ell\\-3k-5\ell\end{bmatrix}$$

so $k + 2\ell = 0$ and $-3k - 5\ell = 1$. From the first equation we obtain $k = -2\ell$, so $\ell = 1$ and $k = -2$. Hence

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = -2T\left(\begin{bmatrix}1\\-3\end{bmatrix}\right) + T\left(\begin{bmatrix}2\\-5\end{bmatrix}\right) = -2\begin{bmatrix}-1\\0\\-1\end{bmatrix} + \begin{bmatrix}-1\\-1\\0\end{bmatrix} = \begin{bmatrix}1\\-1\\2\end{bmatrix}.$$

Finally, we see that $A = \begin{bmatrix}2 & 1\\-3 & -1\\5 & 2\end{bmatrix}$.

3. We have

$$T\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\\1\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\\1\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\-1\\-1\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\0\\-1\end{bmatrix}.$$

Hence $A = \begin{bmatrix}1 & 0 & 0 & -1\\0 & 1 & -1 & 0\\1 & 1 & -1 & -1\end{bmatrix}$. Row-reducing $A$ yields

$$\begin{bmatrix}1 & 0 & 0 & -1\\0 & 1 & -1 & 0\\0 & 1 & -1 & 0\end{bmatrix} \quad \to \quad \begin{bmatrix}1 & 0 & 0 & -1\\0 & 1 & -1 & 0\\0 & 0 & 0 & 0\end{bmatrix}$$

so if $A\underline{x} = \underline{0}$ then $x_4 = t$ and $x_3 = s$ are free variables, $x_2 = s$ and $x_1 = t$. Hence

$$\underline{x} = \begin{bmatrix}t\\s\\s\\t\end{bmatrix} = t\begin{bmatrix}1\\0\\0\\1\end{bmatrix} + s\begin{bmatrix}0\\1\\1\\0\end{bmatrix}$$

so

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is a basis for $\ker(T)$. Also, from the leading 1's in the row-reduced form of $A$, we see that a basis for $\mathrm{im}(T)$ is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Hence the nullity of $T$ and the rank of $T$ are both 2.

4. Let

$$k_1 T(\underline{x}_1) + k_2 T(\underline{x}_2) + \cdots + k_n T(\underline{x}_n) = \underline{0}.$$

We wish to prove that $k_1 = \cdots k_n = 0$. But by the basic properties of linear transformations, we can rewrite this as

$$T(k_1 \underline{x}_1 + k_2 \underline{x}_2 + \cdots + k_n \underline{x}_n) = \underline{0},$$

as because $T$ is one-to-one, we know that $\ker(T) = \{\underline{0}\}$, that is, the only vector which maps to $\underline{0}$ is $\underline{0}$. Hence

$$k_1 \underline{x}_1 + k_2 \underline{x}_2 + \cdots + k_n \underline{x}_n = \underline{0}.$$

But then, because these vectors are assumed to be linearly independent, we have that $k_1 = \cdots = k_n = 0$, as desired.

5. (a) First we check to see if $T$ is one-to-one by determining $\ker(T)$. If $a+b=0$ then $a=-b$. If $b+c=0$ then $b=-c$. If $c+a=0$ then $c=-a$; and hence $b=a$ and so $a=-a$. This implies $a=0$ and therefore $b=c=0$, so $\ker(T) = \{\underline{0}\}$ which means that $T$ is one-to-one.

To see that $T$ is onto, let $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be any vector in $\mathbb{R}^3$. Let $x = a+b$, $y = b+c$, $z = c+a$. Then $b = x-a$ and $c = z-a$ so $y = x+z-2a$ and $a = \frac{1}{2}(x-y+z)$. This tells us that $x = \frac{1}{2}(x-y+z)+b$ so $b = \frac{1}{2}(x+y-z)$, and $z = c + \frac{1}{2}(x-y+z)$ so $c = \frac{1}{2}(-x+y+z)$. In other words,

$$T\left( \begin{bmatrix} \frac{1}{2}(x-y+z) \\ \frac{1}{2}(x+y-z) \\ \frac{1}{2}(-x+y+z) \end{bmatrix} \right) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

for any $x$, $y$, $z$, and so $T$ is onto. Hence $T$ is an isomorphism.

(b) Recall that the zero vector in $P_3$ is the polynomial with only zero coefficients. Then $a+b=0$, $d=0$, $c=0$ and $a-b=0$. The first and fourth equations tells us that $a=b=0$ as well, so $\ker(T) = \{\underline{0}\}$ and thus $T$ is one-to-one.

Let $k_0 + k_1 x + k_2 x^2 + k_3 x^3$ be any vector in $P_3$. Then we set $a+b = k_0$, $d = k_1$, $c = k_2$ and $a-b = k_3$. Adding the first and fourth equations yields $a = \frac{1}{2}(k_0+k_3)$ while substracting them tells us that $b = \frac{1}{2}(k_0 = k_3)$. Hence

$$T\left(\begin{bmatrix} \frac{1}{2}(k_0 + k_3) & \frac{1}{2}(k_0 - k_3) \\ k_2 & k_1 \end{bmatrix}\right) = k_0 + k_1x + k_2x^2 + k_3x^3$$

for any coefficients $k_0$, $k_1$, $k_2$, $k_3$, and thus $T$ is onto. We can now conclude that it is an isomorphism.

6. If $p(x) = k_0 + k_1x + k_2x^2$ then

$$ST(p(x)) = S(k_1 + k_2x + k_0x^2) = k_1 + (k_0 + k_1 + k_2)x + (4k_0 + k_1 + 2k_2)x^2$$

and

$$TS(p(x)) = T(k_0+(k_0+k_1+k_2)x+(k_0+2k_1+4k_2)x^2) = (k_0+k_1+k_2)+(k_0+2k_1+4k_2)x+k_0x^2.$$

7. Since $S$ is one-to-one, the only vector mapped to $\underline{0}$ (in $U$) is $\underline{0}$ (in $W$). But since $T$ is one-to-one, the only vector mapped to $\underline{0}$ (in $W$) is $\underline{0}$ (in $V$). Thus the only vector that $ST$ maps to $\underline{0}$ (in $U$) is $\underline{0}$ (in $V$), that is, $\ker(ST) = \{\underline{0}\}$. Hence $ST$ is one-to-one.

8. To show that $T$ is invertible, we must show that it is an isomorphism. If $T$ maps to the zero vector then we have $a - c = 0$, $b - d = 0$, $2a - c = 0$ and $2b - d = 0$. The first and third equations tell us that $a = 2a$ and so $a = c = 0$. Similarly, the second and fourth equations tell us that $b = d = 0$. Thus $\ker(T) = \{\underline{0}\}$ and so $T$ is one-to-one.

   Next, let $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ be any vector in $M_{22}$. Then we set $a - c = x$, $b - d = y$, $2a - c = z$, $2b - d = w$. Subtracting the first equation from the third equation, we see that $a = z - x$ and so $c = z - 2x$. Subtracting the second equation from the fourth equation, we see that $b = w - y$ and so $d = w - 2y$. Hence

$$T\left(\begin{bmatrix} z - x & w - y \\ z - 2x & w - 2y \end{bmatrix}\right) = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

   for any $x$, $y$, $z$, $w$. Thus $T$ is onto and therefore is an isomorphism. This also tells us that

$$T^{-1}\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right) = \begin{bmatrix} z - x & w - y \\ z - 2x & w - 2y \end{bmatrix}.$$