MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 8 **Mathematics 2051** FALL 2007

SOLUTIONS

[4] 1. We know that

$$\dim(U + \dim(U^{\perp}) = n.$$

The set $\{\underline{f}_1, \ldots, \underline{f}_p\}$ is linearly independent (because it is orthogonal), and so this is a basis for U; hence $\dim(U) = p$. Thus $\dim(U^{\perp}) = n - p$, and indeed there are $n - p$ vectors in $\{\underline{f}_{p+1}, \ldots, \underline{f}_n\}$. But note that this set is in $U^{\perp}$ because $\{\underline{f}_1, \ldots, \underline{f}_n\}$ is an orthogonal set, and so each of the vectors in $\{\underline{f}_{p+1}, \ldots, \underline{f}_n\}$ is orthogonal to all of the vectors in $\{\underline{f}_1, \ldots, \underline{f}_p\}$, and thus is orthogonal to all of U. Furthermore, the set $\{\underline{f}_{p+1}, \ldots, \underline{f}_n\}$ is also orthogonal, so these vectors are linearly independent. Since any set of $n - p$ vectors in $U^{=}perp$ must be a basis for $U^{\perp}$, we conclude that

$$U^{\perp} = \operatorname{span}\{\underline{f}_{p+1}, \ldots, \underline{f}_n\}.$$

[3] 2. (a) The eigenvalues of A are $\lambda_1 = 14$ and $\lambda_2 = 4$. The matrix $A - 14I$ is

$$\begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

so $x_2 = t$ is a free variable and $x_1 = 3t$. Thus an eigenvector corresponding to λ_1 is

$$\underline{x}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

The matrix $A - 4I$ is

$$\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{bmatrix}$$

so $x_2 = t$ is a free variable and $x_1 = -\frac{1}{3}t$. Thus an eigenvector corresponding to λ_2 is

$$\underline{x}_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$

Since these eigenvectors correspond to distinct eigenvalues, they must be orthogonal. Normalising them, then, we see that

$$Q = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 14 & 0 \\ 0 & 4 \end{bmatrix}.$$

[4] (b) The eigenvalues of A are $\lambda_1 = -2$ (multiplicity 2) and $\lambda_2 = 8$. The matrix $A + 2I$ is

$$\begin{bmatrix} 5 & 0 & 5 \\ 0 & 0 & 0 \\ 5 & 0 & 5 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so $x_3 = t$ and $x_2 = s$ are free variables, and $x_1 = -t$. Thus

$$\underset{\sim}{x} = \begin{bmatrix} -t \\ s \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

and so two linearly independent eigenvectors corresponding to λ_1 are

$$\underset{\sim}{x}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \underset{\sim}{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Note that, although it was not guaranteed, these eigenvectors are orthogonal.
The matrix $A = 8I$ is

$$\begin{bmatrix} -5 & 0 & 5 \\ 0 & -10 & 0 \\ 5 & 0 & -5 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so $x_3 = t$ is a free variable, $x_2 = 0$ and $x_1 = t$. Thus an eigenvector corresponding to λ_2 is

$$\underset{\sim}{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

This must be orthogonal to the other eigenvectors, because they correspond to distinct eigenvalues.
Hence normalisation gives

$$Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 8 & 0 & 0 \end{bmatrix}.$$

[6] (c) The eigenvalues of A are $\lambda_1 = -3$ (multiplicity 2) and $\lambda_2 = 6$. The matrix $A + 3I$ is

$$\begin{bmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -2 & 1 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & -1 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so $x_3 = t$ and $x_2 = s$ are free variables, and $x_1 = s - \frac{1}{2}t$. Thus

$$\underset{\sim}{x} = \begin{bmatrix} s - \frac{1}{2}t \\ s \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

and so

$$\underset{\sim}{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \quad \text{and} \quad \underset{\sim}{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

are linearly independent eigenvectors corresponding to λ_1. However, these are not orthogonal; we need to find an orthogonal basis for the same eigenspace using the Gram-Schmidt algorithm. We let $\hat{\underset{\sim}{x}}_1 = \underset{\sim}{x}_1$ and so

$$\hat{\underset{\sim}{x}}_2 = \underset{\sim}{x}_2 - \frac{\underset{\sim}{x}_2 \cdot \hat{\underset{\sim}{x}}_1}{\|\hat{\underset{\sim}{x}}_1\|^2}\hat{\underset{\sim}{x}}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ 1 \\ \frac{2}{5} \end{bmatrix}.$$

Now $\hat{\underset{\sim}{x}}_1$ and $\hat{\underset{\sim}{x}}_2$ are orthogonal eigenvectors corresponding to λ_1.
The matrix $A - 6I$ is

$$\begin{bmatrix} -5 & -4 & 2 \\ -4 & -5 & -2 \\ 2 & -2 & -8 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & \frac{4}{5} & -\frac{2}{5} \\ 0 & -\frac{9}{5} & -\frac{18}{5} \\ 0 & -\frac{18}{5} & -\frac{36}{5} \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & \frac{4}{5} & -\frac{2}{5} \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

so $x_3 = t$ is a free variable, $x_2 = -2t$ and $x_1 = 2t$. Hence

$$\underset{\sim}{x}_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

is an eigenvector corresponding to λ_3.
Normalising these, we get

$$Q = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} & \frac{2}{3} \\ 0 & \frac{5}{3\sqrt{5}} & -\frac{2}{3} \\ -\frac{2}{\sqrt{5}} & \frac{2}{3\sqrt{5}} & \frac{1}{3} \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

[3] 3. Recall that for any square matrix A, $\det(A) = \det(A^T)$ and $\det(A^{-1}) = \frac{1}{\det(A)}$. Since Q is orthogonal, $Q^T = Q^{-1}$, and thus we have

$$\begin{aligned} \det(Q) &= \frac{1}{\det(Q)} \\ [\det(Q)]^2 &= 1 \\ \det(Q) &= \pm 1. \end{aligned}$$

[2] 4. (a) The eigenvalues of A are $\lambda_1 = 2$ (multiplicity 2) and $\lambda_2 = 8$. Since these are both positive, A is positive definite.

[3] (b) We row-reduce A using only the operation of adding a multiple of one row to another row:

$$\begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 4 & 2 & 2 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 4 & 2 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & \frac{8}{3} \end{bmatrix}$$

and then we divide each row of this matrix by the square root of its diagonal element to get

$$U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & \sqrt{3} & \frac{1}{\sqrt{3}} \\ 0 & 0 & \frac{2\sqrt{6}}{3} \end{bmatrix}.$$

[3] 5. By definition,

$$\begin{aligned} \|4\underline{x} + 3\underline{y}\|^2 &= \langle 4\underline{x} + 3\underline{y}, 4\underline{x} + 3\underline{y} \rangle \\ &= \langle 4\underline{x}, 4\underline{x} \rangle + \langle 4\underline{x}, 3\underline{y} \rangle + \langle 3\underline{y}, 4\underline{x} \rangle + \langle 3\underline{y}, 3\underline{y} \rangle \\ &= 16\langle \underline{x}, \underline{x} \rangle + 12\langle \underline{x}, \underline{y} \rangle + 12\langle \underline{y}, \underline{x} \rangle + 9\langle \underline{y}, \underline{y} \rangle \\ &= 16\|\underline{x}\|^2 + 24\langle \underline{x}, \underline{y} \rangle + 9\|\underline{y}\|^2 \\ &= 16(2^2) + 24(-5) + 9(5^2) \\ &= 169. \end{aligned}$$

In other words, this is precisely the same question given on Assignment 7, just recast in the context of inner products.

[4] 6. (a) Let

$$\underline{x} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad \underline{y} = \begin{bmatrix} r & s \\ t & u \end{bmatrix}$$

be two vector in M_{22}. Then

$$\begin{aligned} T(\underline{x} + \underline{y}) &= T\left(\begin{bmatrix} a+r & b+s \\ c+t & d+u \end{bmatrix} \right) \\ &= \begin{bmatrix} (a+r) - (d+u) \\ 2(b+s) \\ c+t \end{bmatrix} \\ &= \begin{bmatrix} (a-d) + (r-u) \\ 2b + 2s \\ c+t \end{bmatrix} \\ &= \begin{bmatrix} a-d \\ 2b \\ c \end{bmatrix} + \begin{bmatrix} r-u \\ 2s \\ t \end{bmatrix} \\ &= T(\underline{x}) + T(\underline{y}). \end{aligned}$$

Also, for any scalar k,

$$\begin{aligned} T(k\underline{x}) &= T\left(\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}\right) \\ &= \begin{bmatrix} ka - kd \\ 2kb \\ kc \end{bmatrix} \\ &= k\begin{bmatrix} a - d \\ 2b \\ c \end{bmatrix} \\ &= kT(\underline{x}). \end{aligned}$$

Hence T is a linear transformation.

[2] (b) Note that the zero vector in P_2 is $\underline{0} = 0$, but

$$T(\underline{0}) = 1 + x + x^2.$$

Since $T(\underline{0}) \neq \underline{0}$, T is not a linear transformation.

[2] (c) If A and B are matrices in M_{nn}, we have $T(A + B) = \det(A + B)$. In general, however, $\det(A + B) \neq \det(A) + \det(B) = T(A) + T(B)$. Hence T is not a linear transformation.

[4] (d) Let $p(x) = a + bx + cx^2$ and $q(x) = d + ex + fx^2$ be two vectors in P_2. Then

$$\begin{aligned} T(p(x) + q(x)) &= T[(a + d) + (b + e)x + (c + f)x^2] \\ &= (a + d)x + (b + e)x^2 + (c + f)x^3 \\ &= (ax + bx^2 + cx^3) + (d + ex + fx^2) \\ &= T(p(x)) + T(q(x)). \end{aligned}$$

Also, for any scalar k,

$$\begin{aligned} T(kp(x)) &= T(ka + kbx + kcx^2) \\ &= kax + kbx^2 + kcx^3 \\ &= k(ax + bx^2 + cx^3) \\ &= kT(p(x)). \end{aligned}$$

Thus T is a linear transformation.