Assignment 7 | **Mathematics 2051** | Fall 2007

SOLUTIONS

[3] 1. (a) We must simply verify that each pair of vectors is orthogonal:

$$\underline{x}_1 \cdot \underline{x}_2 = (1)(2) + (-3)(0) + (0)(-4) + (2)(-1) = 2 + 0 + 0 - 2 = 0$$
$$\underline{x}_1 \cdot \underline{x}_3 = (1)(1) + (-3)(-1) + (0)(1) + (2)(-2) = 1 + 3 + 0 - 4 = 0$$
$$\underline{x}_2 \cdot \underline{x}_3 = (2)(1) + (0)(-1) + (-4)(1) + (-1)(-2) = 2 + 0 - 4 + 2 = 0.$$

Since these vectors are orthogonal, they are linearly independent, and we are given that they span U; hence they are an orthogonal basis for U.

[3] (b) We must simply normalize the given vectors. Observe that

$$\|\underline{x}_1\| = \sqrt{1^2 + (-3)^2 + 0^2 + 2^2} = \sqrt{14}$$
$$\|\underline{x}_2\| = \sqrt{2^2 + 0^2 + (-4)^2 + (-1)^2} = \sqrt{21}$$
$$\|\underline{x}_3\| = \sqrt{1^2 + (-1)^2 + 1^2 + (-2)^2} = \sqrt{7}$$

so an orthonormal basis for U is the set

$$\left\{ \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \end{bmatrix}, \quad \frac{1}{\sqrt{21}} \begin{bmatrix} 2 \\ 0 \\ -4 \\ -1 \end{bmatrix}, \quad \frac{1}{\sqrt{7}} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -2 \end{bmatrix} \right\}.$$

[4] (c) We use the Expansion Theorem. Observe that

$$\frac{\underline{y} \cdot \underline{x}_1}{\|\underline{x}_1\|^2} = \frac{-6}{14} = -\frac{3}{7}$$
$$\frac{\underline{y} \cdot \underline{x}_2}{\|\underline{x}_2\|^2} = \frac{12}{21} = \frac{4}{7}$$
$$\frac{\underline{y} \cdot \underline{x}_3}{\|\underline{x}_3\|^2} = \frac{30}{7}$$

so then

$$\underline{y} = -\frac{3}{7} \begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \end{bmatrix} + \frac{4}{7} \begin{bmatrix} 2 \\ 0 \\ -4 \\ -1 \end{bmatrix} + \frac{30}{7} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -2 \end{bmatrix}.$$

[5] (d) Note that

$$\frac{\underline{z} \cdot \underline{x}_1}{\|\underline{x}_1\|^2} = \frac{14}{14} = 1$$
$$\frac{\underline{z} \cdot \underline{x}_2}{\|\underline{x}_2\|^2} = \frac{-9}{21} = -\frac{3}{7}$$
$$\frac{\underline{z} \cdot \underline{x}_3}{\|\underline{x}_3\|^2} = \frac{-10}{7}$$

so

$$\underline{p} = \begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \end{bmatrix} - \frac{3}{7}\begin{bmatrix} 2 \\ 0 \\ -4 \\ -1 \end{bmatrix} - \frac{10}{7}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -9 \\ -11\ 2 \\ 37 \end{bmatrix}.$$

Note that

$$\underline{z} - \underline{p} = \frac{1}{7}\begin{bmatrix} -5 \\ -3 \\ -2 \\ -2 \end{bmatrix}$$

is a vector in $U^\perp$, and so

$$\underline{z} = \frac{1}{7}\begin{bmatrix} -9 \\ -11\ 2 \\ 37 \end{bmatrix} + \frac{1}{7}\begin{bmatrix} -5 \\ -3 \\ -2 \\ -2 \end{bmatrix}.$$

[5] (e) We can find a basis for $U^\perp$ by finding a basis for the null space of the corresponding matrix A whose rows are the basis vectors of U. Then

$$A = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 2 & 0 & -4 & -1 \\ 1 & -1 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 6 & -4 & -5 \\ 0 & 2 & 1 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & -\frac{2}{3} & -\frac{5}{6} \\ 0 & 0 & \frac{7}{3} & -\frac{7}{3} \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & -\frac{2}{3} & -\frac{5}{6} \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

so the only free variable is $x_4 = t$, and $x_3 = t$, $x_2 = \frac{3}{2}t$, $x_1 = \frac{5}{2}t$. Hence a basis for $U^\perp$ is

$$\left\{ \begin{bmatrix} 5 \\ 3 \\ 2 \\ 2 \end{bmatrix} \right\}$$

and $\dim(U^\perp) = 1$.

Alternatively, we can recall that

$$\dim(U) + \dim(U^\perp) = n,$$

where here we already know that $n = 4$ and $\dim(U) = 3$, implying that $\dim(U^\perp) = 1$. As such, any vector in $U^\perp$ is a basis vector, such as the one found in part (c), which is simply the preceding basis vector multiplied by $-\frac{1}{7}$.

[5] 2. By the definition of vector length,

$$\begin{aligned}
\|4\underline{x} + 3\underline{y}\|^2 &= (4\underline{x} + 3\underline{y}) \cdot (4\underline{x} + 3\underline{y}) \\
&= 16\underline{x} \cdot \underline{x} + 24\underline{x} \cdot \underline{y} + 9\underline{y} \cdot \underline{y} \\
&= 16\|\underline{x}\|^2 + 24\underline{x} \cdot \underline{y} + 9\|\underline{y}\|^2 \\
&= 16(2^2) + 24(-5) + 9(5^2) \\
&= 169.
\end{aligned}$$

[11] 3. We let

$$\underline{f}_1 = \underline{x}_1 = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix},$$

so then

$$\begin{aligned}
\underline{f}_2 &= \underline{x}_2 - \frac{\underline{x}_2 \cdot \underline{f}_1}{\|\underline{f}_1\|^2}\underline{f}_1 \\
&= \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix} - \frac{7}{10}\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} \\
&= \frac{1}{10}\begin{bmatrix} -27 \\ -9 \\ 40 \end{bmatrix}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\underline{f}_3 &= \underline{x}_3 - \frac{\underline{x}_3 \cdot \underline{f}_1}{\|\underline{f}_1\|^2}\underline{f}_1 - \frac{\underline{x}_3 \cdot \underline{f}_2}{\|\underline{f}_2\|^2}\underline{f}_2 \\
&= \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix} - \frac{4}{10}\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} - \frac{-\frac{134}{5}}{\frac{241}{10}} \cdot \frac{1}{10}\begin{bmatrix} -27 \\ -9 \\ 40 \end{bmatrix} \\
&= \frac{1}{241}\begin{bmatrix} 144 \\ 48 \\ 108 \end{bmatrix}.
\end{aligned}$$

Hence the desired orthogonal basis is

$$\left\{ \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \ \frac{1}{10}\begin{bmatrix} -27 \\ -9 \\ 40 \end{bmatrix}, \ \frac{1}{241}\begin{bmatrix} 144 \\ 48 \\ 108 \end{bmatrix} \right\}.$$

[4] 4. We simply project $\underline{y}$ onto the subspace spanned by $\underline{x}_1$ and $\underline{x}_2$ and obtain the resulting vector $\underline{p} = \text{proj}_U \underline{y}$:

$$\underline{p} = \frac{-6}{36}\begin{bmatrix}-4\\-4\\2\\0\end{bmatrix} + \frac{-11}{6}\begin{bmatrix}-1\\0\\-2\\-1\end{bmatrix} = \frac{1}{6}\begin{bmatrix}15\\4\\20\\11\end{bmatrix}.$$

This is the vector in U closest to $\underline{y}$.