MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 6 **Mathematics 2051** FALL 2007

SOLUTIONS

[5] 1. (a) We set

$$\underline{0} = k_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} + k_3 \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} + k_4 \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} k_1 + 3k_3 + 2k_4 & k_2 + 3k_4 \\ 2k_3 + k_4 & 2k_2 + k_3 + 4k_4 \end{bmatrix}.$$

This results in four equations:

$$\begin{aligned} k_1 + 3k_3 + 2k_4 &= 0 \\ k_2 + 3k_4 &= 0 \\ 2k_3 + k_4 &= 0 \\ 2k_2 + k_3 + 4k_4 &= 0. \end{aligned}$$

Writing this system as a matrix and row-reducing, we obtain

$$\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 2 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & -\frac{5}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

so $k_1 = k_2 = k_3 = k_4 = 0$. Hence U is linearly independent.

[5] (b) We set

$$\underline{0} = k_1(x^3 + 2x) + k_2(3 - x^3) + k_3(2x^2 - 2x + 1)$$
$$= (k_1 - k_2)x^3 + 2k_3x^2 + (2k_1 - 2k_3)x + (3k_2 + k_3).$$

The coefficient of each power of x must be zero, which provides the system

$$\begin{aligned} k_1 - k_2 &= 0 \\ 2k_3 &= 0 \\ 2k_1 - 2k_3 &= 0 \\ 3k_2 + k_3 &= 0. \end{aligned}$$

The second equation immediately indicates that $k_3 = 0$ and so, from the third and fourth equation, we also see that $k_1 = k_2 = 0$. Hence U is linearly independent.

[5] (c) We have

$$\begin{aligned}
\underline{0} &= \frac{k_1}{x^2-4} + \frac{k_2}{x^2+x-2} + \frac{k_3}{x^2-3x+2} \\
&= \frac{k_1}{(x-2)(x+2)} + \frac{k_2}{(x+2)(x-1)} + \frac{k_3}{(x-2)(x-1)} \\
&= \frac{k_1(x-1)+k_2(x-2)+k_3(x+2)}{(x-2)(x+2)(x-1)} \\
&= \frac{(k_1+k_2+k_3)x+(-k_1-2k_2+2k_3)}{(x-2)(x+2)(x-1)}.
\end{aligned}$$

This implies that

$$\begin{aligned}
k_1+k_2+k_3 &= 0 \\
-k_1-2k_2+2k_3 &= 0.
\end{aligned}$$

The corresponding matrix is

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & 2 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 3 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \end{bmatrix}$$

so $x_3 = t$ is a free variable, $x_2 = 3t$ and $x_1 = -4t$. Hence U is linearly dependent; for instance,

$$\frac{-4}{x^2-4} + \frac{3}{x^2+x-2} + \frac{1}{x^2-3x+2} = 0.$$

[4] 2. Note that we assume that A and B are non-zero matrices, as otherwise they are clearly not linearly independent. As usual, we set

$$kA + \ell B = \underline{0},$$

and we wish to prove that $k = \ell = 0$. First note that we can rewrite this equation as

$$kA^T + \ell(-B^T) = kA^T - \ell B^T = \underline{0}$$

since A is symmetric and B is skew-symmetric. But also, note that

$$\begin{aligned}
(kA+\ell B)^T &= \underline{0}^T \\
kA^T + \ell B^T &= \underline{0},
\end{aligned}$$

recalling that the zero vector in this case is the zero matrix, which is its own transpose (that is, it is symmetric as well). Now we have

$$\begin{aligned}
kA^T - \ell B^T &= kA^T + \ell B^T \\
2\ell B^T &= \underline{0}
\end{aligned}$$

and so $\ell = 0$. But now we have that

$$kA^T = \underline{0},$$

and thus $k = 0$ as well. Hence $\{A, B\}$ is linearly independent.

[5] 3. (a) Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a general vector in M_{22}. Then

$$AX = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ -c & -d \end{bmatrix}$$

while

$$XA = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} a & -b \\ c & -d \end{bmatrix}.$$

If $AX = XA$ then this tells us that

$$\begin{aligned} a &= a \\ b &= -b \\ -c &= c \\ -d &= -d. \end{aligned}$$

Obviously, the first and fourth equations provide no information but we see that $b = c = 0$. Thus a matrix in U is of the form

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

for any scalars a and d. Hence a basis for U is the set

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

and $\dim(U) = 2$.

[5] (b) Let

$$p(x) = ax^3 + bx^2 + cx + d$$

be a general vector in P_3. Then

$$-p(x) = -ax^3 - bx^2 - cx - d$$

while

$$p(-x) = a(-x)^3 + b(-x)^2 + c(-x) + d = -ax^3 - bx^2 - cx - d.$$

Since $-p(x) = p(-x)$, we can equate the coefficients of like powers, so

$$\begin{aligned} -a &= -a \\ -b &= b \\ -c &= -c \\ -d &= d. \end{aligned}$$

In this case, neither the first nor the third equations provide any information, but we can deduce that $b = d = 0$. Hence a vector in U must be of the form

$$p(x) = ax^3 + cx$$

for any scalars a and c. Thus a basis for U is

$$\{x^3, x\}$$

and again $\dim(U) = 2$.

[4] 4. We know that any linearly independent set in a finite dimensional vector space is contained in a basis of that vector space. But any non-zero singleton set (that is, a set containing only one non-zero vector) is necessarily linearly independent, and thus it must be contained in a basis as well.

[4] 5. Let $\dim(U) = p$ then without loss of generality, we can assume that $\{\underline{x}_1, \ldots, \underline{x}_p\}$ is a basis for U. Now, if $\underline{y}$ is in $\text{span}\{\underline{x}_1, \ldots, \underline{x}_n\}$ then

$$U = \text{span}\{\underline{x}_1, \ldots, \underline{x}_n\} = \text{span}\{\underline{y}, \underline{x}_1, \ldots, \underline{x}_n\} = W$$

and hence $\dim(W) = \dim(U)$. Otherwise, if $\underline{y}$ is not in $\text{span}\{\underline{x}_1, \ldots, \underline{x}_n\}$ then $\underline{y}$ is not in $\text{span}\{\underline{x}_1, \ldots, \underline{x}_p\}$ and hence, by the Independence Lemma, $\{\underline{y}, \underline{x}_1, \ldots, \underline{x}_p\}$ is a linearly independent spanning set (that is, a basis) for W so $\dim(W) = \dim(U) + 1$.

[3] 6. Any polynomial is a continuous function on any interval $[a, b]$ so the space P of all polynomials is contained in $F[a, b]$ for any a and b. If there exists a finite basis for $F[a, b]$ then there exists a finite basis for P, which contradicts the fact that P is an infinite dimensional vector space. Hence $F[a, b]$ is also infinite-dimensional.