MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 5 | **Mathematics 2051** | FALL 2007

SOLUTIONS

[5] 1. Assume that there are two zero vectors, $\underline{0}_1$ and $\underline{0}_2$. Then for any vector $\underline{x}$ in the vector space, $\underline{x}+\underline{0}_1=\underline{x}$ and $\underline{x}+\underline{0}_2=\underline{x}$. But then $\underline{x}+\underline{0}_1=\underline{x}+\underline{0}_2$ and so, by the Cancellation Theorem, $\underline{0}_1=\underline{0}_2$. Hence the zero vector is unique.

[5] 2. If $k\underline{x}=\ell\underline{x}$ then $k\underline{x}-\ell\underline{x}=\underline{0}$ and so $(k-\ell)\underline{x}=\underline{0}$. But for scalar p and vector $\underline{y}$, $p\underline{y}=\underline{0}$ implies that $p=0$ or $\underline{y}=\underline{0}$. Here we are given that $\underline{x}\neq\underline{0}$, so it must be that $k-\ell=0$ and thus $k=\ell$.

[5] 3. (a) Observe that the zero matrix (that is, the zero vector in M_{22}) is symmetric, and hence is in U. Also, if A and B are symmetric then

$$(A+B)^T=A^T+B^T=A+B,$$

so U is closed under addition. Finally, for any scalar k,

$$(kA)^T=kA^T=kA,$$

so U is closed under scalar multiplication. Thus U is a subspace of V.

[5] (b) The zero vector is $f(x)\equiv 0$, and

$$\int_0^1 0\,dx=0,$$

so the zero vector is in U. If f and g are in U then

$$\int_0^1 (f+g)(x)\,dx=\int_0^1 [f(x)+g(x)]\,dx=\int_0^1 f(x)\,dx+\int_0^1 g(x)\,dx=0+0=0,$$

so U is closed under addition. If k is any scalar,

$$\int_0^1 (kf)(x)\,dx=\int_0^1 kf(x)\,dx=k\int_0^1 f(x)\,dx=k(0)=0,$$

so U is closed under scalar multiplication. Hence U is a subspace of $F[0,1]$.

[3] (c) From (b), we see that the zero vector is not in U (and, in fact, all three conditions for being a subspace fail). Therefore U is not a subspace of V.

[5] (d) Note that $p(x)=ax^2+bx+c$ so $xp(x)=ax^3+bx^2+cx$. The zero vector can be obtained by setting $a=b=c=0$ so it is in U. If $xp(x)$ and $xq(x)=dx^3+ex^2+fx$ are two vectors in U then

$$\begin{aligned}xp(x)+xq(x)&=(ax^3+bx^2+cx)+(dx^3+ex^2+fx)\\&=(a+d)x^3+(b+e)x^2+(c+f)x\end{aligned}$$

is also in U, which is therefore closed under addition. Finally, if k is a scalar,

$$kxp(x)=k(ax^3+bx^2+cx)=kax^3+kbx^2+kcx$$

is in U so U is closed under scalar multiplication. Hence U is a subspace of P_3.

[2] (e) Here, $p(x) = ax^3 + bx^2 + cx + d$ so $xp(x) = ax^4 + bx^3 + cx^2 + dx$. In general, this is not a member of P_3, and so U is not a subset of P_3; hence it cannot be a subspace of P_3 either.

[5] 4. We set

$$5x^2 - 6x + 7 = k(x^2 - 3) + \ell(3x + 4) = kx^2 + 3\ell x + (4\ell - 3k).$$

Then $k = 5$, $3\ell = -6$ (so $\ell = -2$) and $4\ell - 3k = 7$. However,

$$4(-2) - 3(5) = -23,$$

so the third equation is inconsistent with the first two. Thus $\underline{x}$ is not in the span of $\underline{u}$ and $\underline{v}$.

[5] 5. We need every possible vector in M_{22} to be a linear combination of the given vectors, so for real numbers a, b, c, d we set

$$\begin{aligned}\begin{bmatrix} a & b \\ c & d \end{bmatrix} &= k_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + k_2 \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + k_3 \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} + k_4 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\ &= \begin{bmatrix} k_1 + k_2 + k_3 + k_4 & 2(k_2 + k_3 + k_4) \\ 3(k_3 + k_4) & 4k_4 \end{bmatrix}.\end{aligned}$$

Then we arrive at the system of equations

$$\begin{aligned} k_1 + k_2 + k_3 + k_4 &= a \\ 2(k_2 + k_3 + k_4) &= b \\ 3(k_3 + k_4) &= c \\ 4k_4 &= d. \end{aligned}$$

From the fourth equation, we see that

$$k_4 = \frac{d}{4}.$$

From the third,

$$k_3 = \frac{c}{3} - k_4 = \frac{c}{3} - \frac{d}{4}.$$

From the second,

$$k_2 = \frac{b}{2} - k_3 - k_4 = \frac{b}{2} - \frac{c}{3} + \frac{d}{4} - \frac{d}{4} = \frac{b}{2} - \frac{c}{3}.$$

And from the first,

$$k_1 = a - k_2 - k_3 - k_4 = a - \frac{b}{2} + \frac{c}{3} - \frac{c}{3} + \frac{d}{4} - \frac{d}{4} = a - \frac{b}{2}.$$

Since these exist for any a, b, c, d, we can therefore express any vector in M_{22} as a linear combination of the given vectors. Thus M_{22} is contained in $\text{span}\{\underline{x}_1, \underline{x}_2, \underline{x}_3, \underline{x}_4\}$.

The only other consideration is to note that every vector in $\text{span}\{\underline{x}_1, \underline{x}_2, \underline{x}_3, \underline{x}_4\}$ is clearly a 2×2 matrix, and hence in M_{22}, so $\text{span}\{\underline{x}_1, \underline{x}_2, \underline{x}_3, \underline{x}_4\}$ is contained in M_{22}.

Thus $M_{22} = \text{span}\{\underline{x}_1, \underline{x}_2, \underline{x}_3, \underline{x}_4\}$.