MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 4 — **Mathematics 2051** — FALL 2007

SOLUTIONS

[5] 1. (a) We have

$$\det(A - \lambda I) = -(\lambda^3 - 3\lambda^2 - 6\lambda + 8) = -(\lambda + 2)(\lambda - 1)(\lambda - 4) = 0,$$

so the eigenvalues of A are $\lambda_1 = -2$, $\lambda_2 = 1$ and $\lambda_3 = 4$. Since we have three distinct eigenvalues, we know that A must be diagonalizable.
For $\lambda_1 = -2$, the matrix $A + 2I$ is

$$\begin{bmatrix} 8 & 4 & -4 \\ -4 & -2 & 6 \\ 0 & 0 & 3 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 4 \\ 0 & 0 & 3 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

We see that $x_3 = 0$ and $x_2 = t$ is a free variable, while $x_1 = -\frac{1}{2}t$. Thus the eigenspace corresponding to λ_1 is

$$t \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$$

and so an eigenvector corresponding to λ_1 is $\underline{x}_1 = \left[\begin{smallmatrix} 1 \\ -2 \\ 0 \end{smallmatrix}\right]$.
For $\lambda_2 = 1$, the matrix $A - I$ is

$$\begin{bmatrix} 5 & 4 & -4 \\ -4 & -5 & 6 \\ 0 & 0 & 0 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & \frac{4}{5} & -\frac{4}{5} \\ 0 & -\frac{9}{5} & \frac{14}{5} \\ 0 & 0 & 0 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & \frac{4}{5} & -\frac{4}{5} \\ 0 & 1 & -\frac{14}{9} \\ 0 & 0 & 0 \end{bmatrix}.$$

We see that $x_3 = t$ is a free variable while $x_2 = \frac{14}{9}t$ and $x_1 = -\frac{4}{9}t$. Thus the eigenspace corresponding to λ_2 is

$$t \begin{bmatrix} -\frac{4}{9} \\ \frac{14}{9} \\ 1 \end{bmatrix}$$

and so an eigenvector corresponding to λ_2 is $\underline{x}_2 = \left[\begin{smallmatrix} 4 \\ -14 \\ -9 \end{smallmatrix}\right]$.
For $\lambda_3 = 4$, the matrix $A - 4I$ is

$$\begin{bmatrix} 2 & 4 & -4 \\ -4 & -8 & 6 \\ 0 & 0 & -3 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & 2 & -2 \\ 0 & 0 & -2 \\ 0 & 0 & -3 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & 2 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

We see that $x_3 = 0$ and $x_2 = t$ is a free variable, while $x_1 = -2t$. Thus the eigenspace corresponding to λ_3 is

$$t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

and so an eigenvector corresponding to λ_3 is $\underline{x}_3 = \left[\begin{smallmatrix} -2 \\ 1 \\ 0 \end{smallmatrix}\right]$.

Hence $D = P^{-1}AP$ with

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & 4 & -2 \\ -2 & -14 & 1 \\ 0 & -9 & 0 \end{bmatrix}.$$

[5] (b) We have

$$\det(A - \lambda I) = -(\lambda^3 - \lambda^2 - 5\lambda - 3) = -(\lambda + 1)^2(\lambda - 3) = 0,$$

so the eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 3$. We must check to see if λ_1 has two linearly independent eigenvectors in order to determine whether A is diagonalizable.

For $\lambda_1 = -1$, then, the matrix $A + I$ is

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ -1 & -1 & 1 \end{bmatrix} \quad \longrightarrow \quad \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We see that $x_3 = t$ and $x_2 = s$ are free variables, while $x_1 = t - s$. Thus the eigenspace corresponding to λ_1 is

$$\begin{bmatrix} t - s \\ s \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

Hence λ_1 does indeed have two linearly independent eigenvectors, say $\underline{x}_1 = \left[\begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix}\right]$ and $\underline{x}_2 = \left[\begin{smallmatrix} -1 \\ 1 \\ 0 \end{smallmatrix}\right]$, and therefore A is diagonalizable.

For $\lambda_2 = 3$, the matrix $A - 3I$ is

$$\begin{bmatrix} -3 & 1 & -1 \\ 2 & -2 & -2 \\ -1 & -1 & -3 \end{bmatrix} \quad \longrightarrow \quad \begin{bmatrix} 1 & -\frac{1}{3} & \frac{1}{3} \\ 0 & -\frac{4}{3} & -\frac{8}{3} \\ 0 & -\frac{4}{3} & -\frac{8}{3} \end{bmatrix} \quad \longrightarrow \quad \begin{bmatrix} 1 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

We see that $x_3 = t$ is a free variable, while $x_2 = -2t$ and $x_1 = -t$. Thus the eigenspace corresponding to λ_2 is

$$t \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

and so an eigenvector corresponding to λ_2 is $\underline{x}_3 = \left[\begin{smallmatrix} -1 \\ -2 \\ 1 \end{smallmatrix}\right]$.

Hence $D = P^{-1}AP$ with

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix}.$$

[5] (c) We have

$$\det(A - \lambda I) = -(\lambda^3 - \lambda^2 - \lambda + 1) = -(\lambda - 1)^2(\lambda + 1) = 0,$$

so the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = -1$. Again, we must check to see if λ_1 has two linearly independent eigenvectors in order to determine whether A is diagonalizable. For $\lambda_1 = 1$, then, the matrix $A - I$ is

$$\begin{bmatrix} -1 & 2 & -1 \\ 1 & 0 & -1 \\ 3 & -2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

We see that $x_3 = t$ is a free variables, while $x_2 = t$ and $x_1 = t$ as well. Thus the eigenspace corresponding to λ_1 is

$$t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Hence λ_1 does not possess two linearly independent eigenvectors, and therefore A is not diagonalizable.

[4] 2. Since A and B are similar, there exists an invertible matrix P such that

$$B = P^{-1}AP.$$

But because A is idempotent, we have $A = A^2$. So then

$$B^2 = P^{-1}APP^{-1}AP = P^{-1}A^2P = P^{-1}AP = B,$$

and therefore B is also idempotent.

[4] 3. Note first that $BP^{-1} = P^{-1}A$. If we let $\underline{y} = P^{-1}\underline{x}$ then

$$B\underline{y} = BP^{-1}\underline{x} = P^{-1}A\underline{x} = P^{-1}(\lambda\underline{x}) = \lambda P^{-1}\underline{x} = \lambda\underline{y}.$$

Hence λ is indeed an eigenvalue of B with corresponding eigenvector $\underline{y} = P^{-1}\underline{x}$.

[4] 4. (a) Most axioms will automatically hold by virtue of the properties of 2×2 matrices with the standard operations of matrix addition and scalar multiplication: these includes axioms A2, A3, S2, S3, S4 and S5. For the remaining axioms, let $\underline{x} = \begin{bmatrix} a & a+b \\ a-b & b \end{bmatrix}$ and $\underline{y} = \begin{bmatrix} c & c+d \\ c-d & d \end{bmatrix}$ be any two vectors in A, and let k be a scalar.

First we check axiom A1:

$$\underline{x}+\underline{y}=\begin{bmatrix} a+c & a+b+c+d \\ a-b+c-d & b+d \end{bmatrix}=\begin{bmatrix} a+c & (a+c)+(b+d) \\ (a+c)-(b+d) & b+d \end{bmatrix}$$

so $\underline{x}+\underline{y}$ is in A, and hence A is closed under addition.
For axiom A4, we note that the zero matrix is in A, as can be seen by setting $a=b=0$.
For axiom A5, we note that the negative of $\underline{x}$ is the vector

$$\begin{bmatrix} -a & -(a+b) \\ -(a-b) & -b \end{bmatrix}=\begin{bmatrix} -a & (-a)+(-b) \\ (-a)-(-b) & -b \end{bmatrix}$$

which is also in A.
Finally, for axiom S1, we have

$$k\underline{x}=\begin{bmatrix} ka & k(a+b) \\ k(a-b) & kb \end{bmatrix}=\begin{bmatrix} ka & ka+kb \\ ka-kb & kb \end{bmatrix}$$

which is in A, and thus A is closed under scalar multiplication.
Since all ten axioms hold, A is a vector space.

[3] (b) B does not constitute a vector space: axiom A5 fails (any ordered pair (x,y) with $y>0$ does not have a negative in B) as does axiom S1 (multiplication of any ordered pair (x,y) with $y>0$ by a scalar $k<0$ produces a vector which is not in B).

[3] (c) C does not constitute a vector space: axiom S3 fails. If $\underline{x}=(a,b,c)$ is in C then

$$(k+\ell)\underline{x}=(k+\ell)(a,b,c)=(ka+\ell a,b,c)$$

while

$$k\underline{x}+\ell\underline{x}=k(a,b,c)+\ell(a,b,c)=(ka,b,c)+(\ell a,b,c)=(ka+\ell a,2b,2c)$$

so $(k+\ell)\underline{[}\neq k\underline{x}+\ell\underline{x}$.

[3] (d) D does not constitute a vector space: axioms S3, S4 and S5 all fail. If $\underline{x}=(a,b,c)$ is in D then

$$(k+\ell)\underline{x}=(k+\ell)(a,b,c)=(c,ka+\ell a,b)$$

while

$$k\underline{x}+\ell\underline{x}=k(a,b,c)+\ell(a,b,c)=(c,ka,b)+(c,\ell a,b)=(2c,ka+\ell a,2b)$$

so $(k+\ell)\underline{x}\neq k\underline{x}+\ell\underline{x}$. Also,

$$k(\ell\underline{x})=k[\ell(a,b,c)]=k(c,\ell a,b)=(b,kc,\ell a)$$

while

$$(k\ell)\underline{x}=(k\ell)(a,b,c)=(c,k\ell a,b)$$

so $k(\ell\underline{x})\neq(k\ell)\underline{x}$. Finally,

$$1(a,b,c)=(c,a,b)\neq(a,b,c).$$

[4] (e) Axioms A2, A3, S2, S3, S4 and S5 hold because they hold for all continuous real-valued functions. For the remaining axioms, let f and g be vectors (functions) in E so $f(0) = g(0) = 1$, and take k to be any scalar.

For axiom A1, we see that

$$(f + g)(1) = f(1) + g(1) = 0 + 0 = 0,$$

so $(f + g)$ is in E and thus closure under addition holds.

For axiom A4, observe that the zero vector is the function z such that $z(x) \equiv 0$ for all x, and this is in E.

For axiom A5, note that the negative of f is $-f$ and $(-f)(1) = -f(1) = -0 = 0$ so each negative is in E.

Finally, for axiom S1 we have

$$(kf)(1) = kf(1) = k(0) = 0,$$

so closure under scalar mutliplication is upheld.

Hence E is a vector space.