MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 3 **Mathematics 2051** FALL 2007

SOLUTIONS

1. We row-reduce A:

$$\begin{bmatrix} 1 & -3 & 2 & 5 \\ -3 & 3 & -1 & -9 \\ -2 & 0 & 1 & -4 \\ -4 & -6 & 7 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 2 & 5 \\ 0 & -6 & 5 & 6 \\ 0 & -6 & 5 & 6 \\ 0 & -18 & 15 & 18 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 2 & 5 \\ 0 & 1 & -\frac{5}{6} & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence a basis for $\mathrm{col}(A)$ is

$$\left\{ \begin{bmatrix} 1 \\ -3 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 0 \\ -6 \end{bmatrix} \right\}$$

while a basis for $\mathrm{row}(A)$ is

$$\left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ -1 \\ -9 \end{bmatrix} \right\}.$$

Finally, we conclude that $\mathrm{rank}(A) = 2$.

2. We construct the corresponding matrix A (using the given vectors as the rows of the matrix) and reduce it to row-echelon form:

$$A = \begin{bmatrix} 1 & 0 & -1 & -4 & 2 \\ 1 & -3 & 3 & 3 & 0 \\ 0 & 2 & -2 & -1 & 5 \\ -1 & 2 & -1 & 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & -4 & 2 \\ 0 & -3 & 4 & 7 & -2 \\ 0 & 2 & -2 & -1 & 5 \\ 0 & 2 & -2 & -1 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & -4 & 2 \\ 0 & 1 & -\frac{4}{3} & -\frac{7}{3} & \frac{2}{3} \\ 0 & 0 & \frac{2}{3} & \frac{11}{3} & \frac{11}{3} \\ 0 & 0 & \frac{2}{3} & \frac{11}{3} & \frac{11}{3} \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -1 & -4 & 2 \\ 0 & 1 & -\frac{4}{3} & -\frac{7}{3} & \frac{2}{3} \\ 0 & 0 & 1 & \frac{11}{2} & \frac{11}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We conclude that a basis for U is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 3 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -2 \\ -1 \\ 5 \end{bmatrix} \right\}.$$

3. (a) Reducing A to row-echelon form gives

$$\begin{bmatrix} 1 & 4 & -2 \\ 0 & 3 & 1 \\ 1 & 1 & -3 \\ 2 & -1 & -7 \\ -2 & -5 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & -2 \\ 0 & 3 & 1 \\ 0 & -3 & -1 \\ 0 & -9 & -3 \\ 0 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & -2 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so a basis for $\operatorname{col}(A)$ is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 1 \\ -1 \\ -5 \end{bmatrix} \right\}$$

while a basis for $\operatorname{row}(A)$ is

$$\left\{ \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \right\}.$$

Hence $\operatorname{rank}(A) = 2$.

(b) Our workings from part (a) also tell us that the general solution to the equation $A\underline{x} = \underline{0}$ has free variable $x_3 = t$, while $x_2 = -\frac{1}{3}t$ and $x_1 = \frac{10}{3}t$ so

$$\underline{x} = t \begin{bmatrix} 10 \\ -1 \\ 3 \end{bmatrix}.$$

Hence $\operatorname{null}(A)$ has as its basis the singleton set

$$\left\{ \begin{bmatrix} 10 \\ -1 \\ 3 \end{bmatrix} \right\}$$

(that is, any member of $\operatorname{null}(A)$ is a multiple of the basis vector) and so $\dim[\operatorname{null}(A)] = 1$.

(c) From theory, since A has $n = 3$ columns and $r = \operatorname{rank}(A) = 2$, we expect

$$\dim[\operatorname{null}(A)] = n - r = 3 - 2 = 1,$$

which supports the results of parts (a) and (b).

4. We know that $\operatorname{row}(UA)$ is contained in $\operatorname{row}(A)$, and hence $\dim[\operatorname{row}(UA)] \leq \dim[\operatorname{row}(A)]$. But $\operatorname{rank}(UA) = \dim[\operatorname{row}(UA)]$ and $\operatorname{rank}(A) = \dim[\operatorname{row}(A)]$, so we can conclude that $\operatorname{rank}(UA) \leq \operatorname{rank}(A)$. Furthermore, $\operatorname{row}(UA) = \operatorname{row}(A)$ if U is invertible, so in this case, $\dim[\operatorname{row}(UA)] = \dim[\operatorname{row}(A)]$, and therefore $\operatorname{rank}(UA) = \operatorname{rank}(A)$.

5. (a) Since the columns of A and B are linearly independent, we know that if $A\underline{y} = \underline{0}$ then $\underline{y} = \underline{0}$, and if $B\underline{y} = \underline{0}$ then $\underline{y} = \underline{0}$. So consider any vector $\underline{x}$ such that $AB\underline{x} = \underline{0}$. If we set $B\underline{x} = \underline{y}$ then this becomes $A\underline{y} = \underline{0}$, and hence $\underline{y} = \underline{0}$. But this implies that $B\underline{x} = \underline{0}$, and therefore $\underline{x} = \underline{0}$. Hence the matrix AB has linearly independent columns.

(b) Since the rows of A and B are linearly independent, the columns of A^T and B^T are linearly independent as well. Consider $(AB)^T = B^T A^T$ and let $\underline{x}$ be any vector such that $B^T A^T \underline{x} = \underline{0}$. If we set $A^T \underline{x} = \underline{y}$ then this becomes $B^T \underline{y} = \underline{0}$, which implies that $\underline{y} = \underline{0}$. But now $A^T \underline{x} - \underline{0}$, and thus $\underline{x} = \underline{0}$. We can now conclude that $(AB)^T$ has linearly independent columns, and therefore that the matrix AB has linearly independent rows.