MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Test 2 | **Mathematics 2051** | Fall 2007

SOLUTIONS

[10] 1. We begin by finding the eigenvalues of A. Expanding along the second column, we see that

$$\begin{aligned}\det(A-\lambda I) &= (1-\lambda)[(-5-\lambda)(8-\lambda)-14(-3)]\\ &= (1-\lambda)(\lambda^2-3\lambda+2)\\ &= (1-\lambda)(\lambda-2)(\lambda-1)\\ &= -(\lambda-1)^2(\lambda-2),\end{aligned}$$

so the eigenvalues are $\lambda_1 = 1$ (of multiplicitly 2) and $\lambda_2 = 2$.

Now, $A - \lambda_1 I = A - I$ is the matrix

$$\begin{bmatrix}-6 & 0 & 14\\ -3 & 0 & 7\\ -3 & 0 & 7\end{bmatrix} \to \begin{bmatrix}1 & 0 & -\frac{7}{3}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

so $x_3 = t$ and $x_2 = s$ are free variables, and $x_1 = \frac{7}{3}t$. Hence

$$\underline{x}_1 = \begin{bmatrix}\frac{7}{3}t\\ s\\ t\end{bmatrix} = t\begin{bmatrix}7\\ 0\\ 3\end{bmatrix} + s\begin{bmatrix}0\\ 1\\ 0\end{bmatrix},$$

so

$$\begin{bmatrix}7\\ 0\\ 3\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}0\\ 1\\ 0\end{bmatrix}$$

are two linearly independent eigenvectors corresponding to λ_1.

Next, $A - \lambda_2 I = A - 2I$ is the matrix

$$\begin{bmatrix}-7 & 0 & 14\\ -3 & -1 & 7\\ -3 & 0 & 6\end{bmatrix} \to \begin{bmatrix}1 & 0 & -2\\ 0 & -1 & 1\\ 0 & 0 & 0\end{bmatrix} \to \begin{bmatrix}1 & 0 & -2\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}$$

so $x_3 = t$ is a free variable, while $x_2 = t$ and $x_1 = 2t$. Thus

$$\underline{x}_2 = \begin{bmatrix}2\\ 1\\ 1\end{bmatrix}$$

is an eigenvector corresponding to λ_2.

Since A possesses three linearly independent eigenvectors, it is diagonalizable. Furthermore,

$$P = \begin{bmatrix}7 & 0 & 2\\ 0 & 1 & 1\\ 3 & 0 & 1\end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{bmatrix}.$$

[6] 2. If a matrix B is similar to A then there exists an invertible matrix P such that

$$\begin{aligned} B &= P^{-1}AP \\ &= P^{-1}kIP \\ &= kP^{-1}IP \\ &= kP^{-1}P \\ &= kI \\ &= A, \end{aligned}$$

so the only matrix similar to A is A itself.

[6] 3. If V is a vector space then for scalars k and ℓ and any vector $\underline{x}$ in V, $k(\ell\underline{x}) = (k\ell)\underline{x}$. Here, if $\underline{x} = \begin{bmatrix} a \\ b \end{bmatrix}$,

$$k(\ell\underline{x}) = k\left(\ell\begin{bmatrix} a \\ b \end{bmatrix}\right) = k\begin{bmatrix} 2\ell a \\ 2\ell b \end{bmatrix} = \begin{bmatrix} 2k(2\ell a) \\ 2k(2\ell b) \end{bmatrix} = \begin{bmatrix} 4k\ell a \\ 4k\ell b \end{bmatrix}$$

while

$$(k\ell)\underline{x} = (k\ell)\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2k\ell a \\ 2k\ell b \end{bmatrix}.$$

Thus axiom S4 of the definition of vector spaces does not hold.

As well, for any vector $\underline{x}$ in V, we require that $1\underline{x} = \underline{x}$. However,

$$1\underline{x} = 1\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2(1)a \\ 2(1)b \end{bmatrix} = \begin{bmatrix} 2a \\ 2b \end{bmatrix} \neq \underline{x}.$$

Thus axion S5 also fails.

[9] 4. First we check to see if the zero vector is in U. Recall that the zero vector for $F[0,1]$ is the function $z(x) \equiv 0$. In particular, $z(0) = 0$ and $z(1) = 0$, so $z(0) = z(1)$ as required.

Next, let f and g be two vector in U, so $f(0) = f(1)$ and $g(0) = g(1)$. Then

$$(f+g)(0) = f(0) + g(0) = f(1) + g(1) = (f+g)(1),$$

so U is closed under addition.

Finally, for any scalar k,

$$(kf)(0) = kf(0) = kf(1) = (kf)(1),$$

so U is also closed under scalar multiplication. Hence U is a subspace of $F[0,1]$.

[4] 5. First note that $\dim(\mathbb{R}^4) = 4$. This means if a set is to span $\mathbb{R}^4$ then it must contain at least four vectors, and if a set is to be linearly independent in $\mathbb{R}^4$ then it must contain at most four vectors. The set X, then, cannot span $\mathbb{R}^4$, because it contains too few vectors; hence X is not a basis of $\mathbb{R}^4$. The set Y, meanwhile, cannot be linearly independent because it contains too many vectors; thus Y also is not a basis of $\mathbb{R}^4$.

[8] 6. We set

$$\begin{aligned} a(x^2+3)+b(x-1)+c(2x^2+3x) &= 0 \\ (a+2c)x^2+(b+3c)x+(3a-b) &= 0. \end{aligned}$$

This results in three equations:

$$\begin{aligned} a+2c &= 0 \\ b+3c &= 0 \\ 3a-b &= 0. \end{aligned}$$

Solving the first two equations in terms of c and substituting these into the third equation, we get

$$3(-2c)-(-3c) = -3c = 0,$$

so $c = 0$. Thus $a = b = 0$ as well, and so U is a linearly independent set.

[7] 7. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be a general vector in M_{22}. If it is to be an element of U, we must have

$$\begin{aligned} A^T &= -A \\ \begin{bmatrix} a & c \\ b & d \end{bmatrix} &= \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} \end{aligned}$$

and so

$$\begin{aligned} a &= -a \\ c &= -b \\ b &= -c \\ d &= -d. \end{aligned}$$

This tells us that $a = d = 0$, while $b = t$ is a free variable and $c = -t$. Hence

$$A = \begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix} = t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and so a basis for U is

$$\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}.$$

Lastly, $\dim(U) = 1$.