MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 2 **Mathematics 2051** FALL 2007

SOLUTIONS

[4] 1. (a) Since we have three vectors in $\mathbb{R}^3$, we can simply construct a square matrix A whose columns are the given vectors, and determine if A is invertible. We obtain

$$A = \begin{bmatrix} 4 & 2 & -1 \\ -3 & 0 & 6 \\ 1 & 1 & -4 \end{bmatrix}$$

for which $\det(A) = -33 \neq 0$. Hence this is a linearly independent set of vectors.

[4] (b) Since we have four vectors in $\mathbb{R}^4$, we can again construct a square matrix A:

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -3 & -1 & -4 \\ -1 & 0 & -2 & -4 \\ 5 & 1 & -2 & 9 \end{bmatrix}.$$

Since $\det(A) = 0$, this set of vectors is linearly dependent.

[4] (c) Because we cannot construct a square matrix from these vectors, we instead set a linear combination of the vectors equal to $\underline{0}$:

$$k_1 \begin{bmatrix} 2 \\ -3 \\ -1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 1 \\ 5 \\ 4 \end{bmatrix} + k_3 \begin{bmatrix} 6 \\ -8 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

which yields the equations

$$\begin{aligned} 2k_1 + 6k_3 &= 0 \\ -3k_1 + k_2 - 8k_3 &= 0 \\ -k_1 + 5k_2 + 2k_3 &= 0 \\ 4k_2 + k_3 &= 0. \end{aligned}$$

The first and last equations indicate that $k_1 = -3k_3$ and $k_2 = -\frac{1}{4}k_3$. Substituting this information into the second equation we have

$$-3(-3k_3) - \frac{1}{4}k_3 - 8k_3 = \frac{3}{4}k_3 = 0,$$

so $k_3 = 0$, and thus $k_1 = k_2 = 0$ as well. This is consistent with the third equation, and therefore only the trivial solution results. Hence the vectors are linearly independent.

[4] 2. Again, we can construct a square matrix and find out if $\det(A) = 0$. In this case,

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & -2 & 0 \\ -3 & 1 & -1 \end{bmatrix}$$

so $\det(A) = 8 \neq 0$. (You should recall that because A is a triangular matrix, $\det(A)$ is simply the product of the diagonal elements, making this computation very easy.) Now we can conclude that $\{\underline{x}_1, \underline{x}_2, \underline{x}_3\}$ spans $\mathbb{R}^3$.

[2] 3. (a) Remember that $\text{span}\{\underline{x}-\underline{y},\ 2\underline{x}+3\underline{y}\}$ is the smallest subspace containing the vectors $\underline{x}-\underline{y}$ and $2\underline{x}+3\underline{y}$. In other words, any other subspace which contains these two vectors contains all of $\text{span}\{\underline{x}-\underline{y},\ 2\underline{x}+3\underline{y}\}$. But $\text{span}\{\underline{x}, \underline{y}\}$ is a subspace which contains these two vectors (because they are linear combinations of $\underline{x}$ and $\underline{y}$), and thus $\text{span}\{\underline{x}-\underline{y},\ 2\underline{x}+3\underline{y}\}$ is contained in $\text{span}\{\underline{x}, \underline{y}\}$.

[4] (b) We need to show that $\underline{x}$ and $\underline{y}$ are contained in $\text{span}\{\underline{x}-\underline{y},\ 2\underline{x}+3\underline{y}\}$ by showing that they are linear combinations of the two given vectors. First, set

$$\underline{x} = k(\underline{x}-\underline{y}) + \ell(2\underline{x}+3\underline{y}) = (k+2\ell)\underline{x} + (3\ell-k)\underline{y}.$$

Then we must have $k + 2\ell = 1$ and $3\ell - k = 0$, so $k = \frac{3}{5}$ and $\ell = \frac{1}{5}$. Next, set

$$\underline{y} = k(\underline{x}-\underline{y}) + \ell(2\underline{x}+3\underline{y}) = (k+2\ell)\underline{x} + (3\ell-k)\underline{y}.$$

Then we must have $k + 2\ell = 0$ and $3\ell - k = 1$, so $k = -\frac{2}{5}$ and $\ell = \frac{1}{5}$. Hence both $\underline{x}$ and $\underline{y}$ are in $\text{span}\{\underline{x}-\underline{y},\ 2\underline{x}+3\underline{y}\}$ and so, by the same reasoning as part (a), we know that $\text{span}\{\underline{x}, \underline{y}\}$ is contained in $\text{span}\{\underline{x}-\underline{y},\ 2\underline{x}+3\underline{y}\}$.

[4] 4. (a) We must determine if the given vectors are linearly independent. We set

$$k_1 \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix} + k_2 \begin{bmatrix} 3 \\ 2 \\ -2 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} 1 \\ 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which results in the system

$$\begin{aligned} k_1 + 3k_2 + k_3 &= 0 \\ 2k_2 + 2k_3 &= 0 \\ -k_1 - 2k_2 - 3k_3 &= 0 \\ -2k_1 - 2k_3 &= 0. \end{aligned}$$

From the second and fourth equations we see that $k_1 = k_2 = -k_3$. Substituting this into the first equation we get

$$-k_3 + 3(-k_3) + k_3 = -3k_3 = 0,$$

so $k_3 = 0$ and thus $k_1 = k_2 = 0$. This is consistent with the third equation, so these vectors are linearly independent, and therefore are already a basis for U. Consequently, $\dim(U) = 3$.

[6] (b) Again, we start by determining if the spanning vectors are linearly independent. We have

$$k_1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} + k_2 \begin{bmatrix} -2 \\ 3 \\ 1 \\ -1 \end{bmatrix} + k_3 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \end{bmatrix} + k_4 \begin{bmatrix} -3 \\ 5 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

This time, we'll solve the system by row-reducing the corresponding matrix:

$$\begin{bmatrix} 1 & -2 & 0 & -3 \\ -1 & 3 & 1 & 5 \\ 0 & 1 & 1 & 2 \\ 2 & -1 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 3 & 3 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so we have two free variables, and hence these vectors are not linearly independent.

In fact, because we have two free variables, this means that simply deleting one of the given variables is not enough. To see this, note that we now have $k_4 = t$ and $k_3 = s$ so $k_2 = -s - 2t$ and $k_1 = -2s - t$. Even if we set $t = 0$, we still have $k_3 = s$, $k_2 = -s$ and $k_1 = 2s$, which means that we can construct non-trivial linear combinations of the first three vectors equal to $\underline{0}$. So we actually need to delete two vectors from the given set in order to have linear independence. (If this discussion is not obvious to you, you can also simply delete one of the spanning vectors and then test the remaining three vectors for linear independence in the usual way; the result will be the same.)

Hence a basis for U is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 1 \\ -1 \end{bmatrix} \right\}$$

and $\dim(U) = 2$.

[6] 5. (a) Since $\{\underline{x}, \underline{y}, \underline{z}\}$ is linearly independent, we know that if

$$k_1\underline{x} + k_2\underline{y} + k_3\underline{z} = \underline{0}$$

then $k_1 = k_2 = k_3 = 0$. Now let $\underline{w}$ be any vector not in $\text{span}\{\underline{x}, \underline{y}, \underline{z}\}$ and set

$$\ell_1\underline{w} + \ell_2\underline{x} + \ell_3\underline{y} + \ell_4\underline{z} = \underline{0}.$$

We need to show that ℓ_1, ℓ_2, ℓ_3 and ℓ_4 must all be 0.

First note that if $\ell_1 \neq 0$ then we can write

$$\underline{w} = -\frac{\ell_2}{\ell_1}\underline{x} - \frac{\ell_3}{\ell_1}\underline{y} - \frac{\ell_4}{\ell_1}\underline{z},$$

which expresses $\underline{w}$ as a linear combination of $\underline{x}$, $\underline{y}$ and $\underline{z}$. But we've assumed that this is not the case, so we must have $\ell_1 = 0$. But then we have simply

$$\ell_2\underline{x} + \ell_3\underline{y} + \ell_4\underline{z} = \underline{0},$$

and so $\ell_2 = \ell_3 = \ell_4 = 0$ by the assumption of the linear independence of these three vectors. Hence $\{\underline{w}, \underline{x}, \underline{y}, \underline{z}\}$ is linearly independent as well.

[2] (b) Since $\dim \mathbb{R}^4 = 4$ and we have proved that $\{\underline{w}, \underline{x}, \underline{y}, \underline{z}\}$ is a linearly independent set of 4 vectors, this must also be a basis for $\mathbb{R}^4$.