MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

TEST 1 | **Mathematics 2051** | FALL 2007

SOLUTIONS

[3] 1. (a) A set of vectors $\{\underline{x}_1, \ldots, \underline{x}_p\}$ is linearly independent if

$$k_1\underline{x}_1 + \cdots + k_p\underline{x}_p = \underline{0}$$

has the unique solution $k_1 = \cdots = k_p = 0$.

[3] (b) The span of vectors $\{\underline{x}_1, \ldots, \underline{x}_p\}$ is the set of all linear combinations of those vectors, that is, the set of all vectors of the form

$$k_1\underline{x}_1 + \cdots + k_p\underline{x}_p$$

for scalars $k_1, \ldots, k_p$.

[3] (c) The basis of a subspace U is a linearly independent spanning set for U.

[3] (d) The null space of a matrix A is the set of all vectors $\underline{x}$ such that $A\underline{x} = \underline{0}$.

[6] 2. We must prove that $\text{span}\{\underline{x}, \underline{y}, \underline{z}\} = \text{span}\{\underline{w}, \underline{x}, \underline{y}, \underline{z}\}$. First let $\underline{u}$ be in $\text{span}\{\underline{x}, \underline{y}, \underline{z}\}$, so

$$\underline{u} = k_1\underline{x} + k_2\underline{y} + k_3\underline{z}$$

for scalars k_1, k_2 and k_3. But then we can write

$$\underline{u} = 0\underline{w} + k_1\underline{x} + k_2\underline{y} + k_3\underline{z},$$

so $\underline{u}$ is also a linear combination of $\underline{w}$,$\underline{\text{x}}$, $\underline{y}$, $\underline{z}$, and hence $\underline{u}$ is in $\text{span}\{\underline{w}, \underline{x}, \underline{y}, \underline{z}\}$. Thus $\text{span}\{\underline{x}, \underline{y}, \underline{z}\}$ is contained in $\text{span}\{\underline{w}, \underline{x}, \underline{y}, \underline{z}\}$.

Now let $\underline{u}$ be in $\text{span}\{\underline{w}, \underline{x}, \underline{y}, \underline{z}\}$, so

$$\underline{u} = k_1\underline{w} + k_2\underline{x} + k_3\underline{y} + k_4\underline{z}$$

for scalars k_1, k_2, k_3 and k_4. We are also told that $\underline{w}$ is in $\text{span}\{\underline{x}, \underline{y}, \underline{z}\}$ so we can write

$$\underline{w} = \ell_1\underline{x} + \ell_2\underline{y} + \ell_3\underline{z}$$

for scalars ℓ_1, ℓ_2 and ℓ_3. Thus

$$\begin{aligned}\underline{u} &= k_1(\ell_1\underline{x} + \ell_2\underline{y} + \ell_3\underline{z}) + k_2\underline{x} + k_3\underline{y} + k_4\underline{z} \\ &= (k_1\ell_1 + k_2)\underline{x} + (k_1\ell_2 + k_3)\underline{y} + (k_1\ell_3 + k_4)\underline{z},\end{aligned}$$

and therefore $\underline{u}$ is a linear combination of $\underline{x}$, $\underline{y}$ and $\underline{z}$. Thus $\underline{u}$ is in $\text{span}\{\underline{x}, \underline{y}, \underline{z}\}$, and so $\text{span}\{\underline{w}, \underline{x}, \underline{y}, \underline{z}\}$ is contained in $\text{span}\{\underline{x}, \underline{y}, \underline{z}\}$.

Hence $\text{span}\{\underline{w}, \underline{x}, \underline{y}, \underline{z}\} = \text{span}\{\underline{x}, \underline{y}, \underline{z}\}$, as required.

[10] 3. We set

$$k_1\underline{x}_1 + k_2\underline{x}_2 + k_3\underline{x}_3 = \underline{0},$$

which results in the system of equations

$$\begin{aligned} k_1 + 3k_3 &= 0 \\ -4k_1 + 2k_2 + 2k_3 &= 0 \\ 7k_1 - 3k_2 &= 0. \end{aligned}$$

From the first equation we can write $k_3 = -\frac{1}{3}k_1$. From the third equation we can write $k_2 = \frac{7}{3}k_1$. Substituting both of these into the second equation, we obtain

$$-4k_1 + 2\left(\frac{7}{3}k_1\right) + 2\left(-\frac{1}{3}k_1\right) = 0,$$

so $k_1 = t$ is a free variable while $k_2 = \frac{7}{3}t$ and $k_3 = -\frac{1}{3}t$. We conclude that these vectors are linearly dependent.

Finally, setting $t = 3$, we obtain

$$3\begin{bmatrix} 1 \\ -4 \\ 7 \end{bmatrix} + 7\begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}.$$

[7] 4. First observe that $\underline{0}$ is in null(A) because $A\underline{0} = \underline{0}$. Now let $\underline{x}$ and $\underline{y}$ be vectors in null(A) so $A\underline{x} = A\underline{y} = \underline{0}$. Then

$$A(\underline{x} + \underline{y}) = A\underline{x} + A\underline{y} = \underline{0} + \underline{0} = \underline{0},$$

and hence $\underline{x}+\underline{y}$ is in null(A), which is therefore closed under addition. Finally, for any scalar k,

$$A(k\underline{x}) = kA\underline{x} = k\underline{0} = \underline{0},$$

so $k\underline{x}$ is in null(A) and we can conclude that (A) is closed under scalar multiplication. Thus null(A) is a subspace of $\mathbb{R}^n$.

[7] 5. (a) We begin by row-reducing A:

$$\begin{bmatrix} 1 & 4 & -3 & -8 \\ 0 & -1 & 2 & 4 \\ 1 & 1 & 3 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & -3 & -8 \\ 0 & -1 & 2 & 4 \\ 0 & -3 & 6 & 12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & -3 & -8 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence a basis for the column space is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} \right\}$$

while a basis for the row space is

$$\left\{ \begin{bmatrix} 1 \\ 4 \\ -3 \\ -8 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -1 \\ 2 \\ 4 \end{bmatrix} \right\}.$$

Finally, $\text{rank}(A) = 2$.

[5] (b) From the row-reduced form of A found in part (a), we see that if $A\underline{x} = \underline{0}$ and

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

then $x_4 = t$ and $x_3 = s$ are free variables,

$$x_2 = 2x_3 + 4x_4 = 2s + 4t$$

and

$$x_1 = -4x_2 + 3x_3 + 8x_4 = -5s - 8t.$$

Hence

$$\underline{x} = \begin{bmatrix} -5s - 8t \\ 2s + 4t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -5 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -8 \\ 4 \\ 0 \\ 1 \end{bmatrix},$$

and so a basis for $\text{null}(A)$ is

$$\left\{ \begin{bmatrix} -5 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -8 \\ 4 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Hence $\dim[\text{null}(A)] = 2$.

[3] (c) From class, we expect

$$\dim[\text{null}(A)] = n - r$$

where n is the number of columns of A and $r = \text{rank}(A)$. And indeed, here,

$$n - r = 4 - 2 = 2 = \dim[\text{null}(A)].$$