MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 1 | **Mathematics 2051** | FALL 2007

SOLUTIONS

[2] 1. (a) Since the third component of any vector in U is always 1, the zero vector is not in U. Hence U is not a subspace of $\mathbb{R}^3$.

[4] (b) The zero vector is in U since it can be obtained by setting $x = y = z = 0$. Let $\begin{bmatrix} a \\ b^2 \\ c \end{bmatrix}$ and $\begin{bmatrix} d \\ e^2 \\ f \end{bmatrix}$ be any two vectors in U; we must determine if

$$\begin{bmatrix} a \\ b^2 \\ c \end{bmatrix} + \begin{bmatrix} d \\ e^2 \\ f \end{bmatrix} = \begin{bmatrix} a+d \\ b^2+e^2 \\ c+f \end{bmatrix}$$

is in U. Clearly, $a+d$ and $c+f$ are in $\mathbb{R}$ and since $b^2+e^2 \geq 0$, it must be that $b^2+e^2 = g^2$ for some $g \in \mathbb{R}$. Finally, for any scalar k we must investigate whether

$$k \begin{bmatrix} a \\ b^2 \\ c \end{bmatrix} = \begin{bmatrix} ka \\ kb^2 \\ kc \end{bmatrix}$$

is in U. Now, however, we have a problem: since this must hold for any scalar, it must hold for $k < 0$, in which case kb^2 cannot be the square of a real number. Hence U is not a subspace of $\mathbb{R}^3$.

[4] (c) The zero vector is in U because if $x = y = z = 0$ then this satisfies the equation $2x + 3y = 4z$. Let $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $\begin{bmatrix} d \\ e \\ f \end{bmatrix}$ be vectors in U, with $2a + 3b = 4c$ and $2d + 3e = 4f$. Then

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} a+d \\ b+e \\ c+f \end{bmatrix}$$

and

$$2(a+d) + 3(b+e) = (2a+3b) + (2d+3e) = 4c + 4f = 4(c+f),$$

so U is closed under addition. Lastly, for any scalar k,

$$k \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} ka \\ kb \\ kc \end{bmatrix}$$

for which

$$2(ka) + 3(kb) = k(2a+3b) = k(4c) = 4(kc),$$

so U is also closed under scalar multiplication. Hence U is a subspace of $\mathbb{R}^3$.

[4] (d) The zero vector is in U because it can be obtained by setting $x = y = z = 0$. Now consider vectors $\begin{bmatrix} a+b \\ b+c \\ c+a \end{bmatrix}$ and $\begin{bmatrix} d+e \\ e+f \\ f+a \end{bmatrix}$ in U. Then

$$\begin{bmatrix} a+b \\ b+c \\ c+a \end{bmatrix} + \begin{bmatrix} d+e \\ e+f \\ f+d \end{bmatrix} = \begin{bmatrix} a+b+d+e \\ b+c+e+f \\ a+c+d+f \end{bmatrix} = \begin{bmatrix} (a+d)+(b+e) \\ (b+e)+(c+f) \\ (c+f)+(a+d) \end{bmatrix}.$$

We can see that this is in U by setting $x = a+d$, $y = b+e$ and $z = c+f$. Hence U is closed under addition. Lastly, for any scalar k,

$$k \begin{bmatrix} a+b \\ b+c \\ c+a \end{bmatrix} = \begin{bmatrix} ka+kb \\ kb+kc \\ kc+ka \end{bmatrix},$$

which is seen to be in U by setting $x = ka$, $y = kb$ and $z = kc$. Thus U is closed under scalar multiplication, and so U is a subspace of $\mathbb{R}^3$.

[10] 2. First we show that null A is contained in null(UA). Let $\underline{x}$ be any vector in null(A), so $A\underline{x} = \underline{0}$. Then

$$UA\underline{x} = U(A\underline{x}) = U\underline{0} = \underline{0},$$

so $\underline{x}$ is also in null(UA). Thus null(A) is contained in null(UA).

Next we'll show that null(UA) is contained in null(A). Let $\underline{y}$ be any vector in null(UA), so $UA\underline{y} = \underline{0}$. Then, because U is invertible, we can write

$$\begin{aligned} U^{-1}UA\underline{y} &= U^{-1}\underline{0} \\ IA\underline{y} &= \underline{0} \\ A\underline{y} &= \underline{0}, \end{aligned}$$

which means that $\underline{y}$ is an element of null(A). Thus null(UA) is contained in null(A). But if null(A) is contained in null(UA) and null(UA) is contained in null(A), the only possibility is that null(A) = null(UA).

[8] 3. U consists of all vectors of the form

$$k\underline{x} + \ell\underline{y} = \begin{bmatrix} k+\ell \\ 7k+\ell \\ -4k-3\ell \\ -2k+3\ell \end{bmatrix}.$$

If $\underline{u}$ is in U then this leads to the system of equations

$$\begin{aligned} k+\ell &= -6 \\ 3k+\ell &= 0 \\ -7k-3\ell &= 1 \\ -2k+\ell &= -5. \end{aligned}$$

The second equation implies that $\ell = -3k$ and then the first equations yields $-2k = -6$ so $k = 3$ and thus $\ell = -9$. We must check this against the other two equations: the third equation is $-7(3) - 3(-9) = 6$ as required, but the fourth is $-2(3) + (-9) = -24$, which is definitely not consistent. Hence $\underline{u}$ is not in U.

If $\underline{v}$ is in U then the system of equations becomes

$$\begin{aligned} k + \ell &= 2 \\ 3k + \ell &= -4 \\ -7k - 3\ell &= 6 \\ -2k + \ell &= 11. \end{aligned}$$

From the first and second equations, we see that $2k = -6$ so $k = -3$, and thus $\ell = 2 - k = 5$. Again, we need to verify that the entire system is consistent with this result. The third equation becomes $-7(-3) - 3(5) = 6$ as desired, and the fourth is $-2(-3) + 5 = 11$. Hence $\underline{v}$ is in U and we can write

$$\underline{v} = -3\underline{x} + 5\underline{y}.$$

[8] 4. Any linear combination of $\underline{x}$ and $\underline{y}$ will be of the form

$$k\underline{x} + \ell\underline{y} = \begin{bmatrix} k - \ell \\ 3\ell \\ 2k + 4\ell \end{bmatrix}.$$

Immediately, though, we know that we will have to require that $k - \ell = 0$ so $k = \ell$. Thus we can now write that any vector spanned by $\underline{x}$ and $\underline{y}$ must be of the form $\begin{bmatrix} 0 \\ 3k \\ 6k \end{bmatrix}$. Thus the third component of any such vector will be exactly twice the second, but U exhibits no such restriction. Hence $U \neq \text{span}\{\underline{x}, \underline{y}\}$.