

MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 3.4

Math 2050 Worksheet

WINTER 2018

SOLUTIONS

1. (a) First we find the eigenvalues and eigenvectors of A . We have

$$\det(A - \lambda I) = \begin{vmatrix} 5 - \lambda & -2 \\ 1 & 2 - \lambda \end{vmatrix} = \lambda^2 - 7\lambda + 12 = (\lambda - 4)(\lambda - 3) = 0,$$

so $\lambda_1 = 4$ and $\lambda_2 = 3$. We have two distinct eigenvalues (and hence two linearly independent eigenvectors) for this 2×2 matrix, so A is diagonalisable.

For λ_1 , $A - \lambda I$ is the matrix

$$\begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

so if $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ then $y = t$ and $x = 2t$. Thus an eigenvector corresponding to λ_1 is

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

For λ_2 , $A - \lambda I$ is the matrix

$$\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix}$$

so $y = t$ and $x = t$. Thus an eigenvector corresponding to λ_2 is $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Hence we can let

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}.$$

- (b) We have

$$\det(A - \lambda I) = \begin{vmatrix} 2 - \lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0$$

so $\lambda = 1$ is the only eigenvalue. The matrix $A - \lambda I$ is

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

so $y = t$ and $x = t$. Hence the only corresponding eigenvector is $\mathbf{x} = -11$ (and its multiples). So we do not have the requisite two linearly independent eigenvectors, and therefore A is not diagonalisable.

(c) We have

$$\begin{aligned}\det(A - \lambda I) &= \begin{vmatrix} 2 - \lambda & -16 & -2 \\ 0 & 5 - \lambda & 0 \\ 2 & -8 & -3 - \lambda \end{vmatrix} \\ &= (5 - \lambda)[(2 - \lambda)(-3 - \lambda) + 4] = -\lambda^3 + 4\lambda^2 + 7\lambda - 10 \\ &= (\lambda - 1)((\lambda + 2)(\lambda - 5)) = 0,\end{aligned}$$

so $\lambda_1 = 1$, $\lambda_2 = -2$, $\lambda_3 = 5$. We have three distinct eigenvalues for this 3×3 matrix, so A must be diagonalisable.

For λ_1 , $A - \lambda I$ is

$$\begin{bmatrix} 1 & -16 & -2 \\ 0 & 4 & 0 \\ 2 & -8 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -16 & -2 \\ 0 & 4 & 0 \\ 0 & 24 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -16 & -2 \\ 0 & 1 & 0 \\ 0 & 24 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -16 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so if $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ then $z = t$, $y = 0$ and $x = 2t$. Hence $\mathbf{x}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

For λ_2 , $A - \lambda I$ is

$$\begin{bmatrix} 4 & -16 & -2 \\ 0 & 7 & 0 \\ 2 & -8 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -4 & -\frac{1}{2} \\ 0 & 7 & 0 \\ 2 & -8 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -4 & -\frac{1}{2} \\ 0 & 7 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -4 & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so $z = t$, $y = 0$ and $x = \frac{1}{2}t$. Hence $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

For λ_3 , $A - \lambda I$ is

$$\begin{aligned}\begin{bmatrix} -3 & -16 & -2 \\ 0 & 0 & 0 \\ 2 & -8 & -8 \end{bmatrix} &\longrightarrow \begin{bmatrix} 2 & -8 & -8 \\ 0 & 0 & 0 \\ -3 & -16 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -4 & -4 \\ 0 & 0 & 0 \\ -3 & -16 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -4 & -4 \\ 0 & 0 & 0 \\ 0 & -28 & -14 \end{bmatrix} \\ &\longrightarrow \begin{bmatrix} 1 & -4 & -4 \\ 0 & -28 & -14 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -4 & -4 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}\end{aligned}$$

so $z = t$, $y = -\frac{1}{2}t$ and $x = 2t$. Hence $\mathbf{x}_3 = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$.

Hence we can let

$$P = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 0 & -1 \\ 1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

(d) We have

$$\begin{aligned}\det(A - \lambda I) &= \begin{vmatrix} 8 - \lambda & 9 & -9 \\ 0 & 2 - \lambda & 0 \\ 4 & 6 & -4 - \lambda \end{vmatrix} \\ &= (2 - \lambda)[(8 - \lambda)(-4 - \lambda) + 36] = -\lambda^3 + 6\lambda^2 - 12\lambda + 8 \\ &= -(\lambda - 2)^3 = 0\end{aligned}$$

so $\lambda = 2$ is the only eigenvalue. The matrix $A - \lambda I$ is

$$\begin{bmatrix} 6 & 9 & -9 \\ 0 & 0 & 0 \\ 4 & 6 & -6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & \frac{3}{2} & -\frac{3}{2} \\ 0 & 0 & 0 \\ 4 & 6 & -6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & \frac{3}{2} & -\frac{3}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so $z = t$, $y = s$ and $x = \frac{3}{2}t - \frac{3}{2}s$. Thus we have two linearly independent eigenvectors:

$$\mathbf{x}_1 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}.$$

But we require three such eigenvectors, so A is not diagonalisable.