SOLUTIONS

1. (a) Observe that

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & -1 \\ 5 & -4-\lambda \end{vmatrix} = \lambda^2 + 2\lambda - 3 = (\lambda+3)(\lambda-1) = 0,$$

so $\lambda = -3$ and $\lambda = 1.$ For $\lambda = -3$, $A - \lambda I$ is

$$\begin{bmatrix} 5 & -1 \\ 5 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 5 & -1 \\ 0 & 0 \end{bmatrix}$$

so if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ then $x_2 = t$ is a free variable and $x_1 = \frac{1}{5}x_2 = \frac{1}{5}t$. Thus the eigenvectors corresponding to $\lambda = -3$ are the vectors of the form

$$\mathbf{x} = \begin{bmatrix} \frac{1}{5}t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

For $\lambda = 1$, $A - \lambda I$ is

$$\begin{bmatrix} 1 & -1 \\ 5 & -5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

so $x_2 = t$ is a free variable and $x_1 = x_2 = t$. Hence the eigenvectors corresponding to $\lambda = 1$ are the vectors of the form

$$\mathbf{x} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(b) Observe that

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 2 \\ 6 & 4-\lambda \end{vmatrix} = \lambda^2 - 7\lambda = \lambda(\lambda-7) = 0,$$

so $\lambda = 0$ and $\lambda = 7.$ For $\lambda = 0$, $A - \lambda I$ is

$$\begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & \frac{2}{3} \\ 6 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & \frac{2}{3} \\ 0 & 0 \end{bmatrix}$$

so if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ then $x_2 = t$ is a free variable and $x_1 = -\frac{2}{3}x_2 = -\frac{2}{3}t$. Thus the eigenvectors corresponding to $\lambda = 0$ are the vectors of the form

$$\mathbf{x} = \begin{bmatrix} -\frac{2}{3}t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

For $\lambda = 7$, $A - \lambda I$ is

$$\begin{bmatrix} -4 & 2 \\ 6 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -\frac{1}{2} \\ 6 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

so $x_2 = t$ is a free variable and $x_1 = \frac{1}{2}x_2 = \frac{1}{2}t$. Hence the eigenvectors corresponding to $\lambda = 7$ are the vectors of the form

$$\mathbf{x} = \begin{bmatrix} \frac{1}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

(c) Observe that

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 7-\lambda & 0 & -4 \\ 0 & 5-\lambda & 0 \\ 5 & 0 & -2-\lambda \end{vmatrix} = -\lambda^3 + 10\lambda^2 - 31\lambda + 30 \\ &= -(\lambda - 2)(\lambda - 3)(\lambda - 5) = 0, \end{aligned}$$

where the polynomial can be factored by either long division or synthetic division. So $\lambda = 2$, $\lambda = 3$ and $\lambda = 5$. For $\lambda = 2$, $A - \lambda I$ is

$$\begin{bmatrix} 5 & 0 & -4 \\ 0 & 3 & 0 \\ 5 & 0 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -\frac{4}{5} \\ 0 & 3 & 0 \\ 5 & 0 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -\frac{4}{5} \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -\frac{4}{5} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ then $x_3 = t$ is a free variable, $x_2 = 0$, and $x_1 = \frac{4}{5}x_3 = \frac{4}{5}t$. Thus the eigenvectors corresponding to $\lambda = 2$ are the vectors of the form

$$\mathbf{x} = \begin{bmatrix} \frac{4}{5}t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}.$$

For $\lambda = 3$, $A - \lambda I$ is

$$\begin{bmatrix} 4 & 0 & -4 \\ 0 & 2 & 0 \\ 5 & 0 & -5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 5 & 0 & -5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so $x_3 = t$ is a free variable, $x_2 = 0$, and $x_1 = x_3 = t$. Hence the eigenvectors corresponding to $\lambda = 3$ are the vectors of the form

$$\mathbf{x} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Finally, for $\lambda = 5$, $A - \lambda I$ is

$$\begin{bmatrix} 2 & 0 & -4 \\ 0 & 0 & 0 \\ 5 & 0 & -7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 5 & 0 & -7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so $x_2 = t$ is a free variable, $x_3 = 0$, and $x_1 = 2x_3 = 0$. Hence the eigenvectors corresponding to $\lambda = 5$ are the vectors of the form

$$\mathbf{x} = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

(d) Observe that

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & -1 & 2-\lambda \end{vmatrix} = -\lambda^3 + 5\lambda^2 - 7\lambda + 3 = -(\lambda - 1)^2(\lambda - 3) = 0,$$

so $\lambda = 1$ and $\lambda = 3.$ For $\lambda = 1$, $A - \lambda I$ is

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ then $x_3 = t$ is a free variable, $x_2 = 0$, and $x_1 = -x_3 - x_2 = -t$. Thus the eigenvectors corresponding to $\lambda = 1$ are the vectors of the form

$$\mathbf{x} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

For $\lambda = 3$, $A - \lambda I$ is

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & -2 & 0 \\ 1 & -1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 0 \\ 1 & -1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so $x_3 = t$ is a free variable, $x_2 = 0$, and $x_1 = x_3 + x_2 = t$. Hence the eigenvectors corresponding to $\lambda = 3$ are the vectors of the form

$$\mathbf{x} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

(e) Observe that

$$\det(A-\lambda I)=\begin{vmatrix}2-\lambda & 1 & 1\\ 0 & 3-\lambda & 0\\ 1 & -1 & 2-\lambda\end{vmatrix}=-\lambda^3+7\lambda^2-15\lambda+9=-(\lambda-1)(\lambda-3)^2=0,$$

so $\lambda=1$ and $\lambda=3.$ For $\lambda=1$, $A-\lambda I$ is

$$\begin{bmatrix}1 & 1 & 1\\ 0 & 2 & 0\\ 1 & -1 & 1\end{bmatrix}\longrightarrow\begin{bmatrix}1 & 1 & 1\\ 0 & 2 & 0\\ 0 & 0 & 0\end{bmatrix}\longrightarrow\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$$

so if $\mathbf{x}=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$ then $x_3=t$ is a free variable, $x_2=0$, and $x_1=-x_3-x_2=-t$. Thus the eigenvectors corresponding to $\lambda=1$ are the vectors of the form

$$\mathbf{x}=\begin{bmatrix}-t\\ 0\\ t\end{bmatrix}=t\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}.$$

For $\lambda=3$, $A-\lambda I$ is

$$\begin{bmatrix}-1 & 1 & 1\\ 0 & 0 & 0\\ 1 & -1 & -1\end{bmatrix}\longrightarrow\begin{bmatrix}1 & -1 & -1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

so $x_3=t$ and $x_2=s$ are free variables, and $x_1=x_3+x_2=t+s$. Hence the eigenvectors corresponding to $\lambda=3$ are the vectors of the form

$$\mathbf{x}=\begin{bmatrix}t+s\\ s\\ t\end{bmatrix}=t\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}+s\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}.$$

2. Since $\mathbf{x}$ is an eigenvector of A with corresponding eigenvalue λ, we know that $A\mathbf{x}=\lambda\mathbf{x}$. But since A is invertible, we can multiply by A^{-1} on both sides:

$$\begin{aligned}A^{-1}A\mathbf{x}&=A^{-1}\lambda\mathbf{x}\\ \mathbf{x}&=\lambda A^{-1}\mathbf{x}\\ \frac{1}{\lambda}\mathbf{x}&=A^{-1}\mathbf{x}\\ \mu\mathbf{x}&=A^{-1}\mathbf{x}\end{aligned}$$

where the scalar $\mu=\frac{1}{\lambda}$ exists and is non-zero because $\lambda\neq 0$. Thus $\mathbf{x}$ is an eigenvector of A^{-1} with corresponding eigenvalue $\mu=\frac{1}{\lambda}$.