MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

TEST 2 **MATHEMATICS 2050** WINTER 2026

SOLUTIONS

[10] 1. (a) (i) $A + A^T = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ -1 & 0 \end{bmatrix}$

(ii) We cannot compute AB because the number of columns of A is not equal to the number of rows of B.

(iii) $BA = \begin{bmatrix} 4 & -1 \\ 0 & 3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 11 & -8 \\ 3 & 0 \\ -4 & 4 \end{bmatrix}$

(iv) $A^2 = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 7 & -6 \\ 3 & -2 \end{bmatrix}$

(v) $A^{-1} = \dfrac{1}{0-(-2)} \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} = \dfrac{1}{2} \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$

(vi) We cannot compute B^{-1} because A is not a square matrix.

[6] (b) First we bring A to reduced row-echelon form:

$$\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \xrightarrow{R_2 \to -\frac{1}{2}R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The elementary matrices required to do this are

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}.$$

The associated inverses are

$$E_1^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, \quad E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

and so

$$A = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}.$$

[8] 2. (a) Using Gaussian elimination, we have

$$\begin{bmatrix} 1 & -4 & 2 \\ -2 & 5 & -4 \\ 3 & -7 & 6 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - (-2)R_1} \begin{bmatrix} 1 & -4 & 2 \\ 0 & -3 & 0 \\ 0 & 5 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \to -\frac{1}{3}R_2} \begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 5 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 5R_2} \begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Observe that the third column is not a pivot column, so we set $z = t$. Thus

$$\begin{aligned} z &= t \\ y &= 0 \\ x &= 4y - 2z = -2t. \end{aligned}$$

Hence the solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

[2] (b) We know that the solution of a non-homogeneous system has the form $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$ where $\mathbf{x}_p$ is a particular solution of the non-homogeneous system and $\mathbf{x}_h$ is a solution of the corresponding homogeneous system. Hence

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

[2] (c) Observe that $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are the columns of the matrix which consists of the coefficients of the homogeneous system of equations. Because this system has an infinite number of solutions, there are non-trivial linear combinations of these columns which sum to $\mathbf{0}$. For instance, if $t = 1$ we see that

$$-2\mathbf{v}_1 + 0\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}.$$

Hence these vectors are linearly dependent.

3. Using Gaussian elimination, we obtain

$$\left[\begin{array}{cc|c} 1 & -2 & k \\ 3 & -3 & 1 \\ -1 & 3 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 - (-1)R_1]{R_2 \to R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & -2 & k \\ 0 & 3 & 1-3k \\ 0 & 1 & k \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cc|c} 1 & -2 & k \\ 0 & 1 & k \\ 0 & 3 & 1-3k \end{array}\right] \xrightarrow{R_3 \to R_3 - 3R_2} \left[\begin{array}{cc|c} 1 & -2 & k \\ 0 & 1 & k \\ 0 & 0 & 1-6k \end{array}\right].$$

There will be no solutions if we have a row of zeroes on the left, but a non-zero entry on the right — that is, if $1 - 6k \neq 0$ or, more simply, if $k \neq \frac{1}{6}$.

[6] 4. We have

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 & 1 & 0 \\ 3 & 5 & 7 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - (-1)R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 1 & 0 \\ 0 & -1 & -2 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - (-1)R_2]{R_1 \to R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -5 & -1 & -2 & 0 \\ 0 & 1 & 4 & 1 & 1 & 0 \\ 0 & 0 & 2 & -2 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to \frac{1}{2}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -5 & -1 & -2 & 0 \\ 0 & 1 & 4 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 - 4R_3]{R_1 \to R_1 - (-5)R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -6 & \frac{1}{2} & \frac{5}{2} \\ 0 & 1 & 0 & 5 & -1 & -2 \\ 0 & 0 & 1 & -1 & \frac{1}{2} & \frac{1}{2} \end{array}\right].$$

Thus

$$\boxed{A^{-1} = \begin{bmatrix} -6 & \frac{1}{2} & \frac{5}{2} \\ 5 & -1 & -2 \\ -1 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.}$$