SOLUTIONS

[5]   1. The normal to this plane is given by

$$\mathbf{n} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ -2 \\ -3 \end{bmatrix} = \begin{vmatrix} 0 & 1 \\ -2 & -3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 1 \\ 0 & -3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & 0 \\ 0 & -2 \end{vmatrix} \mathbf{k} = \begin{bmatrix} 2 \\ 9 \\ -6 \end{bmatrix}.$$

Thus the plane has an equation of the form $2x + 9y - 6z = d$. Since the point $P(3, 0, 1)$ lies in the plane, we know that $d = 6 + 0 - 6 = 0$, and so the equation of the plane is

$$2x + 9y - 6z = 0.$$

[2]   2. (a) The direction vector is given by

$$\mathbf{d} = \overrightarrow{QP} = \begin{bmatrix} 6 \\ -2 \\ 8 \end{bmatrix}.$$

Thus a vector equation of the line is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 6 \\ -2 \\ 8 \end{bmatrix}.$$

Note that, in practice, we would want to simplify $\mathbf{d}$ by extracting the common factor of 2, so a better answer would be

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}.$$

[2]   (b) If $(-9, 11, -19)$ lies on ℓ then there must be a value of t for which

$$-9 = 4 + 3t, \quad 11 = 1 - t, \quad -19 = 1 + 4t.$$

Solving the second equation, we find that $t = -10$. Substituting this into the first equation, we have

$$4 + 3t = 4 + 3(-10) = -26 \neq -9.$$

(Similarly, substitution into the third equation also produces an invalid result). Hence there is no solution to this system of equations, and therefore $(-9, 11, -19)$ does not lie on ℓ.

[5] 3. We set
$$1 + 2\alpha = 3 + 4\beta, \quad 1 = -1 - \beta, \quad -6 - 5\alpha = 1 - 4\beta.$$

From the second equation, we see that $\beta = -2$. Substituting this into the first equation, we obtain $\alpha = -3$. Using the values in third equation, we have
$$-6 - 5\alpha = -6 - 5(-3) = 9 \quad \text{and} \quad 1 - 4\beta = 1 - 4(-2) = 9,$$

so this is a solution of the system. Substituting $\alpha = -3$ into the equation of the first line (or $\beta = -2$ into the equation of the second line), we see that $(-5, 1, 9)$ is the point of intersection.

[2] 4. (a) If we set $z = 0$, then from the equation of the second plane, we see that $x = 1$. Substituting both of these values into the equation of the first plane, we obtain $y = 1$, so $(1, 1, 0)$ is such a point.

[4] (b) Since the line of intersection must lie in both planes, its direction vector $\mathbf{d}$ must be orthogonal to the normal of both planes. Since the normal to the first plane is $\begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$

and the normal to the second plane is $\begin{bmatrix} 5 \\ 0 \\ -3 \end{bmatrix}$, we can find $\mathbf{d}$ by setting

$$\mathbf{d} = \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix} \times \begin{bmatrix} 5 \\ 0 \\ -3 \end{bmatrix} = \begin{vmatrix} 4 & 1 \\ 0 & -3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -3 & 1 \\ 5 & -3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -3 & 4 \\ 5 & 0 \end{vmatrix}\mathbf{k} = \begin{bmatrix} -12 \\ -4 \\ -20 \end{bmatrix}.$$

Now we can use the fact that $(1, 1, 0)$ is a point which lies in both planes to deduce that the vector equation of the line is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -12 \\ -4 \\ -20 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}.$$