SOLUTIONS

1. (a) We have

$$
\begin{bmatrix} 4 & -8 & 0 & | & 1 & 0 & 0 \\ 12 & -23 & 0 & | & 0 & 1 & 0 \\ 0 & 20 & 4 & | & 0 & 0 & 1 \end{bmatrix}
\xrightarrow{R_1 \to \frac{1}{4}R_1}
\begin{bmatrix} 1 & -2 & 0 & | & \frac{1}{4} & 0 & 0 \\ 12 & -23 & 0 & | & 0 & 1 & 0 \\ 0 & 20 & 4 & | & 0 & 0 & 1 \end{bmatrix}
$$

$$
\xrightarrow{R_2 \to R_2 - 12R_1}
\begin{bmatrix} 1 & -2 & 0 & | & \frac{1}{4} & 0 & 0 \\ 0 & 1 & 0 & | & -3 & 1 & 0 \\ 0 & 20 & 4 & | & 0 & 0 & 1 \end{bmatrix}
\xrightarrow[R_3 \to R_3 - 20R_2]{R_1 \to R_1 - (-2)R_2}
\begin{bmatrix} 1 & 0 & 0 & | & -\frac{23}{4} & 2 & 0 \\ 0 & 1 & 0 & | & -3 & 1 & 0 \\ 0 & 0 & 4 & | & 60 & -20 & 1 \end{bmatrix}
$$

$$
\xrightarrow{R_3 \to \frac{1}{4}R_3}
\begin{bmatrix} 1 & 0 & 0 & | & -\frac{23}{4} & 2 & 0 \\ 0 & 1 & 0 & | & -3 & 1 & 0 \\ 0 & 0 & 1 & | & 15 & -5 & \frac{1}{4} \end{bmatrix}
$$

so

$$
A^{-1} = \begin{bmatrix} -\frac{23}{4} & 2 & 0 \\ -3 & 1 & 0 \\ 15 & -5 & \frac{1}{4} \end{bmatrix}.
$$

(b) We have

$$
\begin{bmatrix} 1 & 0 & 2 & | & 1 & 0 & 0 \\ 3 & 1 & 0 & | & 0 & 1 & 0 \\ 1 & -1 & 8 & | & 0 & 0 & 1 \end{bmatrix}
\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 3R_1}
\begin{bmatrix} 1 & 0 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & -6 & | & -3 & 1 & 0 \\ 0 & -1 & 6 & | & -1 & 0 & 1 \end{bmatrix}
$$

$$
\xrightarrow{R_3 \to R_3 - (-1)R_2}
\begin{bmatrix} 1 & 0 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & -6 & | & -3 & 1 & 0 \\ 0 & 0 & 0 & | & -4 & 1 & 1 \end{bmatrix}.
$$

Now we have a row of zeros, so there is no way to row-reduce the matrix on the left to I. Hence A is not invertible.

(c) We have

$$
\left[\begin{array}{cccc|cccc}
1 & 0 & 3 & -2 & 1 & 0 & 0 & 0 \\
-4 & 1 & -8 & 8 & 0 & 1 & 0 & 0 \\
6 & 0 & 19 & -12 & 0 & 0 & 1 & 0 \\
0 & -2 & -8 & 1 & 0 & 0 & 0 & 1
\end{array}\right]
$$

$$
\xrightarrow[\substack{R_2 \to R_2-(-4)R_1 \\ R_3 \to R_3-6R_1}]{}
\left[\begin{array}{cccc|cccc}
1 & 0 & 3 & -2 & 1 & 0 & 0 & 0 \\
0 & 1 & 4 & 0 & 4 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & -6 & 0 & 1 & 0 \\
0 & -2 & -8 & 1 & 0 & 0 & 0 & 1
\end{array}\right]
$$

$$
\xrightarrow[R_4 \to R_4-(-2)R_2]{}
\left[\begin{array}{cccc|cccc}
1 & 0 & 3 & -2 & 1 & 0 & 0 & 0 \\
0 & 1 & 4 & 0 & 4 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & -6 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 8 & 2 & 0 & 1
\end{array}\right]
$$

$$
\xrightarrow[\substack{R_1 \to R_1-3R_3 \\ R_2 \to R_2-4R_3}]{}
\left[\begin{array}{cccc|cccc}
1 & 0 & 0 & -2 & 19 & 0 & -3 & 0 \\
0 & 1 & 0 & 0 & 28 & 1 & -4 & 0 \\
0 & 0 & 1 & 0 & -6 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 8 & 2 & 0 & 1
\end{array}\right]
$$

$$
\xrightarrow[R_1 \to R_1-(-2)R_4]{}
\left[\begin{array}{cccc|cccc}
1 & 0 & 0 & 0 & 35 & 4 & -3 & 2 \\
0 & 1 & 0 & 0 & 28 & 1 & -4 & 0 \\
0 & 0 & 1 & 0 & -6 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 8 & 2 & 0 & 1
\end{array}\right]
$$

so then

$$
A^{-1} = \left[\begin{array}{cccc}
35 & 4 & -3 & 2 \\
28 & 1 & -4 & 0 \\
-6 & 0 & 1 & 0 \\
8 & 2 & 0 & 1
\end{array}\right].
$$

2. (a) Writing the system in the form $A\mathbf{x} = \mathbf{b}$ we have

$$
A = \left[\begin{array}{ccc}
1 & 0 & -3 \\
5 & \frac{1}{3} & -15 \\
-1 & 1 & 4
\end{array}\right]
\qquad
\mathbf{x} = \left[\begin{array}{c}
x \\
y \\
z
\end{array}\right]
\qquad
\mathbf{b} = \left[\begin{array}{c}
1 \\
3 \\
-6
\end{array}\right].
$$

First we compute A^{-1}:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & 0 \\ 5 & \frac{1}{3} & -15 & 0 & 1 & 0 \\ -1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3\to R_3\to(-1)R_1]{R_2\to R_2-5R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & -5 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2\to 3R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -15 & 3 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_3\to R_3-R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -15 & 3 & 0 \\ 0 & 0 & 1 & 16 & -3 & 1 \end{array}\right]$$

$$\xrightarrow{R_1\to R_1-(-3)R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 49 & -9 & 3 \\ 0 & 1 & 0 & -15 & 3 & 0 \\ 0 & 0 & 1 & 16 & -3 & 1 \end{array}\right]$$

so

$$A^{-1} = \begin{bmatrix} 49 & -9 & 3 \\ -15 & 3 & 0 \\ 16 & -3 & 1 \end{bmatrix}$$

and hence

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 49 & -9 & 3 \\ -15 & 3 & 0 \\ 16 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 1 \end{bmatrix}.$$

(b) The matrix A in this case would be 3×4 and hence not square. Therefore A would not be invertible.

3. First we try to row-reduce A to I:

$$\begin{bmatrix} 4 & -6 \\ 1 & -1 \end{bmatrix} \xrightarrow{R_1\leftrightarrow R_2} \begin{bmatrix} 1 & -1 \\ 4 & -6 \end{bmatrix}$$

$$\xrightarrow{R_2\to R_2-4R_1} \begin{bmatrix} 1 & -1 \\ 0 & -2 \end{bmatrix} \xrightarrow{R_2\to -\frac{1}{2}R_2} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1\to R_1-(-1)R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then the elementary matrices involved are

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \quad E_4 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The associated inverses are

$$E_1^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \quad E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \quad E_4^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

and so

$$E_4 E_3 E_2 E_1 A = I \quad \Longrightarrow \quad A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$