MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

TEST 1 — **MATH 2050** — WINTER 2013

SOLUTIONS

[3] 1. (a) Since

$$\|\mathbf{u}\| = \sqrt{2^2 + 0^2 + (-1)^2} = \sqrt{5},$$

a unit vector in the same direction as $\mathbf{u}$ is

$$\frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix},$$

and therefore a vector of length 10 in the same direction as $\mathbf{u}$ is

$$10 \cdot \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = 2\sqrt{5}\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}.$$

[5] (b) The normal to such a plane is

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} 0 & 1 \\ -1 & -5 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 4 \\ -1 & -5 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix}\mathbf{k} = \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix}.$$

Thus the equation of the plane has the form

$$x + 6y + 2z = d.$$

Since the plane must include the point $(2, -1, 7)$ we have

$$d = 2 + 6(-1) + 2 \cdot 7 = 10,$$

so the equation of this plane is

$$x + 6y + 2z = 10.$$

[7] (c) We set

$$k_1\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + k_2\begin{bmatrix} 4 \\ 1 \\ -5 \end{bmatrix} + k_3\begin{bmatrix} 6 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

resulting in the system of equations

$$\begin{aligned} 2k_1 + 4k_2 + 6k_3 &= 0 \\ k_2 - k_3 &= 0 \\ -k_1 - 5k_2 &= 0. \end{aligned}$$

From the second equation, $k_3 = k_2$. From the third equation, $k_1 = -5k_2$. Substituting both of these into the first equation yields

$$\begin{aligned} 2(-5k_2) + 4k_2 + 6k_2 &= 0 \\ 0k_2 &= 0, \end{aligned}$$

and so k_2 can assume any value. Hence there are non-trivial solutions to this system, and so these vectors are linearly dependent.

If we set $k_2 = 1$, then $k_1 = -5$ and $k_3 = 1$. This means that

$$-5\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$$

and so we can write

$$\mathbf{w} = 5\mathbf{u} - \mathbf{v}.$$

Alternatively, we also have

$$\mathbf{v} = 5\mathbf{u} - \mathbf{w} \quad \text{or} \quad \mathbf{u} = \frac{1}{5}\mathbf{v} + \frac{1}{5}\mathbf{w}.$$

[4] 2. (a) Let $\mathbf{d}$ be the direction vector of ℓ, and $\mathbf{e}$ be the direction vector of the second line. Thus

$$\mathbf{d} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \implies \|\mathbf{d}\| = \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3}$$

and

$$\mathbf{e} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \implies \|\mathbf{e}\| = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{9} = 3.$$

Then if θ is the angle between the two lines, we know that

$$\begin{aligned} \mathbf{d} \cdot \mathbf{e} &= \|\mathbf{d}\|\|\mathbf{e}\| \cos(\theta) \\ 1 - 2 - 2 &= \sqrt{3} \cdot 3\cos(\theta) \\ \cos(\theta) &= \frac{-3}{3\sqrt{3}} = -\frac{\sqrt{3}}{3}. \end{aligned}$$

[4] (b) For any point (x, y, z) on ℓ, $x = t$, $y = -1 + t$, $z = 3 - t$. If this point also lies on the plane then

$$\begin{aligned} 4t - 2(-1 + t) + (3 - t) &= 8 \\ t &= 3. \end{aligned}$$

Hence ℓ intersects the plane at the point $(3, 2, 0)$.

[7] (c) First we need a vector than begins on ℓ and terminates at P. Since the point $Q(0,-1,3)$ lies on ℓ, such a vector is

$$\mathbf{u} = \overrightarrow{QP} = \begin{bmatrix} 2-0 \\ -1-(-1) \\ -1-3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix}.$$

We project $\mathbf{u}$ onto $\mathbf{d}$ to obtain the vector

$$\mathbf{p} = \operatorname{proj}_{\mathbf{d}} \mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{d}}{\mathbf{d}\cdot\mathbf{d}}\mathbf{d} = \frac{2+0+4}{1+1+1}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}.$$

The vector that represents the straight-line distance from ℓ to P, then, is

$$\mathbf{u}-\mathbf{p} = \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix},$$

and so the distance from ℓ to P is

$$\|\mathbf{u}-\mathbf{p}\| = \sqrt{0^2+(-2)^2+(-2)^2} = \sqrt{8} = 2\sqrt{2}.$$

[4] 3. For $\mathbf{u}$ and $\mathbf{v}$ to be orthogonal, we need

$$\begin{aligned} \mathbf{u}\cdot\mathbf{v} &= 0 \\ x^2+4x-x+2 &= 0 \\ x^2+3x+2 &= 0 \\ (x+2)(x+1) &= 0. \end{aligned}$$

Hence the vectors will be orthogonal if $x=-2$ or $x=-1$.

[6] 4. (a) Vectors $\mathbf{e}$ and $\mathbf{f}$ span a plane π if, for any vector $\mathbf{v}$ in π, there exist scalars k and ℓ such that $\mathbf{v} = k\mathbf{e}+\ell\mathbf{f}$.

(b) Vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\mathbf{u}\cdot\mathbf{v}=0$.

(c) Let $\mathbf{v}$ be any vector in π, so $\mathbf{v} = k\mathbf{e}+\ell\mathbf{f}$ where k and ℓ are scalars. Assume that $\mathbf{u}$ is orthogonal to both $\mathbf{e}$ and $\mathbf{f}$, so $\mathbf{u}\cdot\mathbf{e}=0$ and $\mathbf{u}\cdot\mathbf{f}=0$. Then

$$\begin{aligned} \mathbf{u}\cdot\mathbf{v} &= \mathbf{u}\cdot(k\mathbf{e}+\ell\mathbf{f}) \\ &\mathbf{u}\cdot(k\mathbf{e})+\mathbf{u}\cdot(\ell\mathbf{f}) \\ &= k(\mathbf{u}\cdot\mathbf{e})+\ell(\mathbf{u}\cdot\mathbf{f}) \\ &= k(0)+\ell(0) \\ &= 0. \end{aligned}$$

Hence $\mathbf{u}$ is orthogonal to any such vector $\mathbf{v}$ as well.