MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SOLUTIONS

1. Observe that

$$\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}=3, \quad \mathbf{u}\cdot\mathbf{u}=42, \quad \mathbf{v}\cdot\mathbf{v}=14.$$

Then

$$\operatorname{proj}_{\mathbf{v}}\mathbf{u}=\frac{3}{14}\begin{bmatrix}-3\\-1\\-2\end{bmatrix} \quad\text{and}\quad \operatorname{proj}_{\mathbf{u}}\mathbf{v}=\frac{3}{42}\begin{bmatrix}1\\4\\-5\end{bmatrix}=\frac{1}{14}\begin{bmatrix}1\\4\\-5\end{bmatrix}.$$

2. Consider any point Q on the line, say, $Q(4,3,4)$. Then $\mathbf{u}=\overrightarrow{QP}=\begin{bmatrix}-4\\-4\\-3\end{bmatrix}$. We want the projection $\mathbf{p}$ of $\mathbf{u}$ onto the direction vector $\mathbf{d}$ of the line:

$$\mathbf{p}=\operatorname{proj}_{\mathbf{d}}\mathbf{u}=\frac{-5}{10}\begin{bmatrix}-1\\0\\3\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1\\0\\-3\end{bmatrix}.$$

The distance from P to ℓ will be the length of $\mathbf{u}-\mathbf{p}$. So

$$\mathbf{u}-\mathbf{p}=\frac{1}{2}\begin{bmatrix}-9\\-8\\-3\end{bmatrix} \implies \|\mathbf{u}-\mathbf{p}\|=\frac{1}{2}\sqrt{(-9)^2+(-8)^2+(-3)^2}=\frac{1}{2}\sqrt{154}.$$

3. (a) First we choose a point Q in the plane π, say, $Q(0,4,0)$. The vector $\mathbf{u}$ from P to Q is $\mathbf{u}=\overrightarrow{PQ}=\begin{bmatrix}7\\6\\2\end{bmatrix}$. A normal to the plane is $\mathbf{n}=\begin{bmatrix}-1\\1\\-3\end{bmatrix}$ and we project $\mathbf{u}$ onto $\mathbf{n}$ to obtain $\mathbf{p}$:

$$\mathbf{p}=\operatorname{proj}_{\mathbf{n}}\mathbf{u}=-\frac{7}{11}\begin{bmatrix}-1\\1\\-3\end{bmatrix}=\frac{7}{11}\begin{bmatrix}1\\-1\\3\end{bmatrix}.$$

Then the distance from the point to π is

$$\|\mathbf{p}\|=\frac{7}{11}\sqrt{1^2+(-1)^2+3^2}=\frac{7\sqrt{11}}{11}.$$

(b) Let $R(x, y, z)$ be the point in π closest to P. Then R is the terminating point of the vector $\mathbf{p} = \overrightarrow{PR}$, and we have that

$$\frac{7}{11}\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} x+7 \\ y+2 \\ z+2 \end{bmatrix}$$

so $x = -\frac{70}{11}$, $y = -\frac{29}{11}$ and $z = -\frac{1}{11}$: the point closest to P is $(-\frac{70}{11}, -\frac{29}{11}, -\frac{1}{11})$.

4. (a) Any vector in the plane will obey the equation $z = 6x - y$ so a vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is of the form

$$\begin{bmatrix} x \\ y \\ 6x-y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 6 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},$$

and so two vectors in the plane are $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 6 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. These are not orthogonal, but we can project $\mathbf{u}$ onto $\mathbf{v}$ to obtain the vector $\mathbf{p}$:

$$\mathbf{p} = \text{proj}_{\mathbf{v}}\,\mathbf{u} = \frac{-6}{2}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \implies \mathbf{u} - \mathbf{p} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$$

and this is orthogonal to $\mathbf{v}$. Thus a pair of orthogonal vectors in π are $\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

(b) From part (a), two orthogonal vectors in the plane are $\mathbf{e} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ and $\mathbf{f} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$. So then

$$\text{proj}_{\pi}\,\mathbf{w} = \frac{5}{2}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \frac{33}{19}\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} = \frac{1}{38}\begin{bmatrix} 66 \\ 293 \\ 103 \end{bmatrix}.$$