Section 3.5: Similarity and Diagonalisation

Defn: Matrices A and B are similar if there exists an invertible matrix P for which $P^{-1}AP = B$.

eg Given $A = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -11 & -5 \\ 31 & 14 \end{bmatrix}$,

consider $P = \begin{bmatrix} 9 & 4 \\ 2 & 1 \end{bmatrix}$. Then $P^{-1} = \begin{bmatrix} 1 & -4 \\ -2 & 9 \end{bmatrix}$.

Then $P^{-1}AP = \begin{bmatrix} 1 & -4 \\ -2 & 9 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 9 & 4 \\ 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} -1 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 9 & 4 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & -5 \\ 31 & 14 \end{bmatrix} = B$$

Thus A and B are similar.

Two similar matrices share many characteristics.

Theorem: If A and B are similar then $\det(A) = \det(B)$.

Proof: We assume that A and B are similar, so there exists an invertible matrix P for which

$$P^{-1}AP = B$$
$$\det(P^{-1}AP) = \det(B)$$
$$\det(P^{-1})\det(A)\det(P) = \det(B)$$
$$\frac{1}{\det(P)}\det(A)\det(P) = \det(B)$$
$$\frac{\det(P)}{\det(P)}\det(A) = \det(B)$$
$$\boxed{\det(A) = \det(B)}$$

As a result, even the eigenvalues of similar matrices must be equal.

Def'n: A matrix A is <u>diagonalisable</u> if it is similar to a diagonal matrix D.

To see how to diagonalise a matrix A, consider the matrix P with columns containing the eigenvectors $\underline{x}_1, \underline{x}_2, \ldots \underline{x}_n$ of A. Then

$$AP = A \begin{bmatrix} \underline{x}_1 & \underline{x}_2 & \cdots & \underline{x}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$= \begin{bmatrix} A\underline{x}_1 & A\underline{x}_2 & \cdots & A\underline{x}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1\underline{x}_1 & \lambda_2\underline{x}_2 & \cdots & \lambda_n\underline{x}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$= \begin{bmatrix} \underline{x}_1 & \underline{x}_2 & \cdots & \underline{x}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$$= PD$$

where D is a diagonal matrix whose entries are the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A.

eg For $A = \begin{bmatrix} -1 & 1 \\ 5 & 3 \end{bmatrix}$ we found that A has

eigenvalues $\lambda_1 = 4$ and $\lambda_2 = -2$ with corresponding eigenvectors $\underline{x}_1 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ and $\underline{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Thus A is similar to $D = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$ and we have that $P = \begin{bmatrix} 1 & -1 \\ 5 & 1 \end{bmatrix}$ so $P^{-1}AP = D$.

Thus A is diagonalisable.

The matrices D and P are not unique. We can order the eigenvalues however we want to make D, and choose any appropriate eigenvector for the corresponding column of P.

<u>Theorem</u>: An $n \times n$ matrix is diagonalisable if and only if it has n linearly independent eigenvectors.

eg For $A = \begin{bmatrix} 2 & -1 & 8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$

We have seen that A has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 2$

with corresponding eigenvectors $\underline{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\underline{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

However we need 3 linearly independent eigenvectors but we only have 2, so A is <u>not</u> diagonalisable.

eg For $A = \begin{bmatrix} 2 & 1 & -1 \\ -4 & 6 & -2 \\ 2 & -1 & 5 \end{bmatrix}$

we have seen that A has eigenvalues $\lambda_1 = 5$ and $\lambda_2 = 4$.

The only eigenvector corresponding to λ_1 is $\underline{x}_1 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$.

However, both $\underline{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\underline{x}_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$ are

eigenvectors corresponding to λ_2.

Hence A <u>is</u> diagonalisable with

$$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} -1 & 1 & -1 \\ -2 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

<u>Theorem</u>: Eigenvectors corresponding to different eigenvalues must be linearly independent.

Proof: First suppose we have a matrix A with two eigenvalues λ_1 and λ_2. Suppose $\underline{x}_1$ and $\underline{x}_2$ are the corresponding eigenvectors. We set

$$k_1 \underline{x}_1 + k_2 \underline{x}_2 = \underline{0}$$

and we want to show that $k_1 = k_2 = 0$.

Then we can write

$$A(k_1 \underline{x}_1 + k_2 \underline{x}_2) = A\underline{0}$$

$$A(k_1 \underline{x}_1) + A(k_2 \underline{x}_2) = \underline{0}$$

$$k_1 (A\underline{x}_1) + k_2 (A\underline{x}_2) = \underline{0}$$

$$k_1 (\lambda_1 \underline{x}_1) + k_2 (\lambda_2 \underline{x}_2) = \underline{0}$$

$$(k_1 \lambda_1) \underline{x}_1 + (k_2 \lambda_2) \underline{x}_2 = \underline{0}$$

Alternatively we could multiply the equation by λ_1 to get

$$\lambda_1 (k_1 \underline{x}_1 + k_2 \underline{x}_2) = \lambda_1 \underline{0}$$

$$(k_1 \lambda_1) \underline{x}_1 + (k_2 \lambda_1) \underline{x}_2 = \underline{0}$$

Thus $(k_1 \lambda_1) \underline{x}_1 + (k_2 \lambda_2) \underline{x}_2 = (k_1 \lambda_1) \underline{x}_1 + (k_2 \lambda_1) \underline{x}_2$

$$(k_2 \lambda_2) \underline{x}_2 = (k_2 \lambda_1) \underline{x}_2$$

$$(k_2 \lambda_2) \underline{x}_2 - (k_2 \lambda_1) \underline{x}_2 = \underline{0}$$

$$k_2 (\lambda_2 - \lambda_1) \underline{x}_2 = \underline{0}$$

Since we know that $\underline{x}_2$ is non-zero (because it's an eigenvector) and we have $\lambda_2 - \lambda_1 \neq 0$ (because they're assumed to be different eigenvalues) it must be that $k_2 = 0$.

Likewise, if we multiply the original equation by λ_2 we would see that $k_1 = 0$.

Thus $\underline{x}_1$ and $\underline{x}_2$ must be linearly independent. We can then generalise the same argument to show that this must also be true for n eigenvalues.

eg Show that A is diagonalisable where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ -4 & 1 & -7 & 0 \\ 0 & 2 & -2 & -1 \end{bmatrix}$$

Here A is lower triangular so the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 3$, $\lambda_3 = -7$, $\lambda_4 = -1$.

Because this 4×4 matrix has 4 distinct eigenvalues, it must have 4 linearly independent eigenvectors. Thus A <u>is</u> diagonalisable.

Diagonal matrices are convenient for many reasons such as computing powers. Given

$$D = \begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix}$$

Then, for any natural number k,

$$D^k = \begin{bmatrix} d_1^k & & & 0 \\ & d_2^k & & \\ & & \ddots & \\ 0 & & & d_n^k \end{bmatrix}$$

If we have a diagonalisable matrix A, we can write

$$\begin{aligned} P^{-1}AP &= D \\ AP &= PD \\ A &= PDP^{-1} \end{aligned}$$

so then
$$\begin{aligned} A^k &= (PDP^{-1})^k \\ &= PDP^{-1}PDP^{-1}PDP^{-1} \cdots PDP^{-1} \\ &= PDIDIDI \cdots IDP^{-1} \\ &= PDDD \cdots DP^{-1} \\ &= PD^kP^{-1} \end{aligned}$$

eg Given $A = \begin{bmatrix} -1 & 1 \\ 5 & 3 \end{bmatrix}$ we have shown that A is diagonalisable with $D = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & -1 \\ 5 & 1 \end{bmatrix}$.

Find A^6.

Then $P^{-1} = \frac{1}{1-(-5)} \begin{bmatrix} 1 & 1 \\ -5 & 1 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 & 1 \\ -5 & 1 \end{bmatrix}$.

So
$$
\begin{aligned}
A^6 &= [PDP^{-1}]^6 \\
&= \begin{bmatrix} 1 & -1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}^6 \frac{1}{6} \begin{bmatrix} 1 & 1 \\ -5 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 4096 & 0 \\ 0 & 64 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 1 & 1 \\ -5 & 1 \end{bmatrix} \\
&= \frac{1}{6} \begin{bmatrix} 4096 & -64 \\ 20160 & 64 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -5 & 1 \end{bmatrix} \\
&= \boxed{\begin{bmatrix} 736 & 672 \\ 3360 & 3424 \end{bmatrix}}
\end{aligned}
$$

— THE END —