Section 3.4: The Eigenvalues and Eigenvectors of a Matrix

Def'n: Let A be a square matrix. A number λ is called an eigenvalue of A if

$$A\underline{x} = \lambda \underline{x}$$

for some non-zero vector $\underline{x}$. Furthermore, each $\underline{x}$ is called an eigenvector of A corresponding to λ. The set of all eigenvectors together with $\underline{0}$ comprise the eigenspace of A corresponding to λ.

eg Let $A = \begin{bmatrix} -1 & 1 \\ 5 & 3 \end{bmatrix}$ and $\underline{x} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$. Then

$$A\underline{x} = \begin{bmatrix} -1 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 20 \end{bmatrix}$$

$$= 4\begin{bmatrix} 1 \\ 5 \end{bmatrix} = 4\underline{x}$$

Thus $\lambda = 4$ is an eigenvalue of A, with eigenvector $\underline{x} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

Given that we want $A\underline{x} = \lambda \underline{x}$ we can rearrange this equation to get

$$A\underline{x} - \lambda \underline{x} = \underline{0}$$
$$A\underline{x} - \lambda I \underline{x} = \underline{0}$$
$$(A - \lambda I)\underline{x} = \underline{0}$$

To avoid the possibility that $\underline{x} = \underline{0}$, we require that $A - \lambda I$ is <u>not</u> invertible. Hence

$$\det(A - \lambda I) = 0$$

where $\det(A - \lambda I)$ is the <u>characteristic polynomial</u> of A.

To find the eigenvalues and eigenvectors of A, then:

1. Identify the characteristic polynomial
2. Find its roots; these are the eigenvalues
3. For each eigenvalue, solve the homogeneous system $(A - \lambda I)\underline{x} = \underline{0}$; each solution is a corresponding eigenvector.

eg $A = \begin{bmatrix} -1 & 1 \\ 5 & 3 \end{bmatrix}$ $\quad A - \lambda I = \begin{bmatrix} -1-\lambda & 1 \\ 5 & 3-\lambda \end{bmatrix}$

$$\det(A - \lambda I) = (-1-\lambda)(3-\lambda) - 5$$
$$= \lambda^2 - 2\lambda - 8$$

The characteristic polynomial is $\lambda^2 - 2\lambda - 8$.

We set $\lambda^2 - 2\lambda - 8 = 0$

$$(\lambda-4)(\lambda+2) = 0$$

The eigenvalues are $\lambda = 4$ and $\lambda = -2$.

For $\lambda = 4$, $A - \lambda I = \begin{bmatrix} -5 & 1 \\ 5 & -1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - (-1)R_1} \begin{bmatrix} -5 & 1 \\ 0 & 0 \end{bmatrix}$

If $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ then $x_2 = t$ and $-5x_1 + x_2 = 0$

$$x_1 = \frac{1}{5}t$$

so the eigenvectors have the form $\underline{x} = \begin{bmatrix} \frac{1}{5}t \\ t \end{bmatrix} = t \begin{bmatrix} 1/5 \\ 1 \end{bmatrix}$

$$= t \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

For $\lambda = -2$, $A - \lambda I = \begin{bmatrix} 1 & 1 \\ 5 & 5 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 5R_1} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

so $x_2 = t$ and $x_1 + x_2 = 0$

$$x_1 = -t$$

Thus the eigenvectors have the form $\underline{x} = \begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Eigenvalues are often complex numbers.

eg $A = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}$ $\quad A - \lambda I = \begin{bmatrix} 1-\lambda & 4 \\ -1 & 1-\lambda \end{bmatrix}$

$$\det(A - \lambda I) = (1-\lambda)^2 + 4$$
$$= \lambda^2 - 2\lambda + 5$$

We set $\lambda^2 - 2\lambda + 5 = 0$

$$\lambda = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2}$$
$$= 1 \pm 2i$$

For $\lambda = 1 + 2i$,

$$A - \lambda I = \begin{bmatrix} -2i & 4 \\ -1 & -2i \end{bmatrix} \xrightarrow{R_2 \to R_2 - (\frac{1}{2i})R_1} \begin{bmatrix} -2i & 4 \\ 0 & 0 \end{bmatrix}$$

$-2i - \frac{4}{2i}$

$= -2i - \frac{2}{i} \cdot \frac{i}{i}$

$= -2i - \frac{2i}{-1} = 0$

Then $x_2 = t$ and $-2i x_1 + 4x_2 = 0$

$$-2i x_1 = -4t$$
$$x_1 = \frac{4}{2i} t = -2it$$

The corresponding eigenvectors are $\underline{x} = \begin{bmatrix} -2it \\ t \end{bmatrix} = t \begin{bmatrix} -2i \\ 1 \end{bmatrix}$.

For $\lambda = 1 - 2i$,

$$A - \lambda I = \begin{bmatrix} 2i & 4 \\ -1 & 2i \end{bmatrix} \xrightarrow{R_2 \to R_2 - (-\frac{1}{2i})R_1} \begin{bmatrix} 2i & 4 \\ 0 & 0 \end{bmatrix}$$

Then $x_2 = t$ and $2i x_1 + 4x_2 = 0$ so $x_1 = 2it$ so the eigenvectors have the form $\underline{x} = \begin{bmatrix} 2it \\ t \end{bmatrix} = t \begin{bmatrix} 2i \\ 1 \end{bmatrix}$.

Theorem: If a matrix is triangular, its diagonal elements are also its eigenvalues.

eg $A = \begin{bmatrix} 2 & -1 & 8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$ Immediately we see that the eigenvalues are $\lambda = 2$ and $\lambda = 0$ because A is upper triangular.

For $\lambda = 0$, $A - \lambda I = \begin{bmatrix} 2 & -1 & 8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & -1 & 8 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

$\longrightarrow \begin{bmatrix} 2 & -1 & 8 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Here $x_2 = t$

$x_3 = 0$

$2x_1 - x_2 + 8x_3 = 0$

$2x_1 = t - 0 \rightarrow x_1 = \frac{1}{2}t$

The corresponding eigenvector is $\underline{x} = \begin{bmatrix} \frac{1}{2}t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

For $\lambda = 2$,

$A - \lambda I = \begin{bmatrix} 0 & -1 & 8 \\ 0 & -2 & 6 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & -8 \\ 0 & -2 & 6 \\ 0 & 0 & 0 \end{bmatrix}$

$\longrightarrow \begin{bmatrix} 0 & 1 & -8 \\ 0 & 0 & -10 \\ 0 & 0 & 0 \end{bmatrix}$

Here $x_1 = t$

$x_3 = 0$

$x_2 - 8x_3 = 0 \rightarrow x_2 = 0$

The corresponding eigenvector is $\underline{x} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

eg $A = \begin{bmatrix} 2 & 1 & -1 \\ -4 & 6 & -2 \\ 2 & -1 & 5 \end{bmatrix}$ $\quad A - \lambda I = \begin{bmatrix} 2-\lambda & 1 & -1 \\ -4 & 6-\lambda & -2 \\ 2 & -1 & 5-\lambda \end{bmatrix}$

$$\det(A-\lambda I) = -\lambda^3 + 13\lambda^2 - 56\lambda + 80$$
$$= -(\lambda-4)^2(\lambda-5)$$

The eigenvalues are $\lambda = 5$ and $\lambda = 4$.

For $\lambda = 5$,

$$A - \lambda I = \begin{bmatrix} -3 & 1 & -1 \\ -4 & 1 & -2 \\ 2 & -1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1/3 & 1/3 \\ -4 & 1 & -2 \\ 2 & -1 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -1/3 & 1/3 \\ 0 & -1/3 & -2/3 \\ 0 & -1/3 & -2/3 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -1/3 & 1/3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -1/3 & 1/3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Here $x_3 = t$

$x_2 = -2x_3 = -2t$

$x_1 = \frac{1}{3}x_2 - \frac{1}{3}x_3$

$\quad = -\frac{2}{3}t - \frac{1}{3}t$

$\quad = -t$

The eigenvector corresponding to $\lambda = 5$ is

$$\begin{bmatrix} -t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}.$$

For $\lambda = 4$,

$$A - \lambda I = \begin{bmatrix} -2 & 1 & -1 \\ -4 & 2 & -2 \\ 2 & -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1/2 & 1/2 \\ -4 & 2 & -2 \\ 2 & -1 & 1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -1/2 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Here $x_3 = t$

$x_2 = s$

$x_1 = \frac{1}{2}x_2 - \frac{1}{2}x_3 = \frac{1}{2}s - \frac{1}{2}t$

The eigenvector corresponding to $\lambda = 4$ is

$$\begin{bmatrix} \frac{1}{2}s - \frac{1}{2}t \\ s \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2}s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2}t \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

The eigenspace for $\lambda = 4$ consists of all linear combinations of the linearly independent eigenvectors $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$