Section 3.3: Complex Numbers

The integers $\mathbb{Z}$ are the positive and negative whole numbers and zero: $0, \pm 1, \pm 2, \pm 3, \ldots$ They allow us to solve simple equations like $x+3=0$.

The rational numbers $\mathbb{Q}$ are quotients of the form $\frac{a}{b}$ where a, b are integers and $b \neq 0$. They solve equations like $4x-5=0$.

The irrational numbers are solutions to equations like $x^2-2=0$.

Together, the rational and irrational numbers comprise the real numbers $\mathbb{R}$.

But even the real numbers don't allow us to solve an equation like $x^2+1=0$.

We need to define a set of numbers beyond the real numbers for which $x^2+1=0$ has a solution.

Thus we define the number i such that $i^2 = -1$.

Now we can define the complex numbers $\mathbb{C}$ to consist of all numbers of the form $z = \alpha + i\beta$ where α and β are real numbers.

We call α the real part of z and β the ~~complex~~ imaginary part of z. If $\alpha = 0$ then $z = i\beta$ is called an imaginary number.

eg Solve $x^2 + 4x + 5 = 0$.

$$x = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1} = \frac{-4 \pm \sqrt{-4}}{2}$$

$$= \frac{-4 \pm \sqrt{4} \cdot \sqrt{-1}}{2}$$

$$= \frac{-4 \pm 2i}{2}$$

$$\boxed{= -2 \pm i}$$

Complex numbers can be added by adding their real parts and adding their imaginary parts; likewise for subtraction.

eg $(1+2i) + (3+4i) = (1+3) + (2+4)i \boxed{= 4 + 6i}$

Complex numbers can be multiplied like linear polynomials, with the fact that $i^2 = -1$.

eg $(1+2i)(3+4i) = 1 \cdot 3 + 1 \cdot 4i + 2i \cdot 3 + 2i \cdot 4i$

$$= 3 + 4i + 6i - 8$$

$$\boxed{= -5 + 10i}$$

Two complex numbers are equal if their real parts are equal and their imaginary parts are equal.

Thus if $\alpha + i\beta = 0$ then $\alpha = \beta = 0$.

If $z = \alpha + i\beta$ then its reciprocal is given by

$$\frac{1}{z} = \frac{1}{\alpha + i\beta}$$

We can simplify this by multiplying by the conjugate $\alpha - i\beta$. Then we have

$$\frac{1}{z} = \frac{1}{\alpha + i\beta} \cdot \frac{\alpha - i\beta}{\alpha - i\beta}$$

$$= \frac{\alpha - i\beta}{\alpha^2 - i\alpha\beta + i\alpha\beta + \beta^2}$$

$$= \frac{\alpha - i\beta}{\alpha^2 + \beta^2} = \frac{\alpha}{\alpha^2 + \beta^2} - i\,\frac{\beta}{\alpha^2 + \beta^2}$$

Note that we typically denote the conjugate of z as $\bar{z}$, so $\bar{z} = \alpha - i\beta$.

We can divide two complex numbers w and z by multiplying both by the conjugate of the denominator z, that is, $\bar{z}$.

$$\text{eg} \quad \frac{2+2i}{2-3i} = \frac{2+2i}{2-3i} \cdot \frac{2+3i}{2+3i}$$

$$= \frac{4+6i+4i-6}{4+6i-6i+9}$$

$$= \frac{-2+10i}{13} \quad \boxed{= -\frac{2}{13} + \frac{10}{13} i}$$

Here we use the fact that if $z = \alpha + i\beta$ then

$$z\bar{z} = (\alpha + i\beta)(\alpha - i\beta)$$

$$= \alpha^2 - i\alpha\beta + i\alpha\beta + \beta^2$$

$$= \alpha^2 + \beta^2$$

We can also use this to define the absolute value or <u>modulus</u> of a complex number:

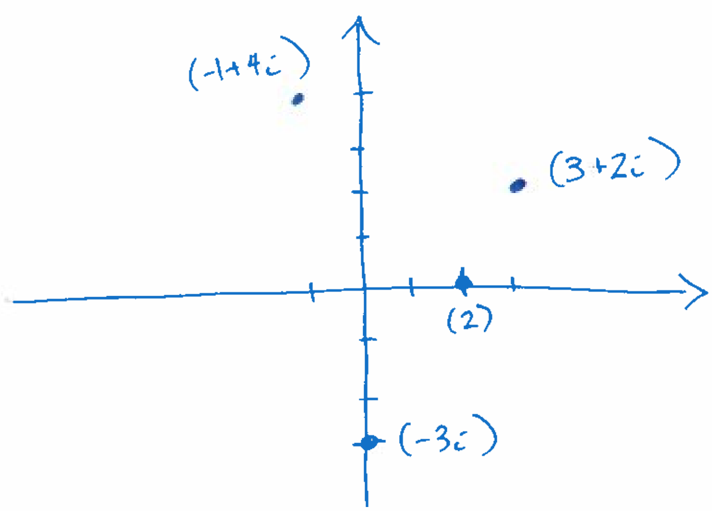
$$|z| = \sqrt{z\bar{z}} = \sqrt{\alpha^2 + \beta^2}$$

There are many elementary properties regarding the conjugate and modulus of complex numbers:

1. $\overline{z \pm w} = \overline{z} \pm \overline{w}$
2. $\overline{zw} = \overline{z}\,\overline{w}$
3. $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$
4. $\overline{(\overline{z})} = z$
5. z is real if and only if $z = \overline{z}$
6. $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$
7. $|z| \geq 0$ for all z
8. $|z| = 0$ if and only if $z = 0$
9. $|zw| = |z||w|$
10. $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$
11. $|z+w| \leq |z| + |w|$

Complex numbers are plotted on the <u>complex plane</u>, which consists of a horizontal axis called the <u>real axis</u> and a vertical axis called the <u>imaginary axis</u>. Then we plot the complex number $z = \alpha + i\beta$ as the point with coordinates (α, β) in the complex plane.

eg



An n-dimensional vector can have complex components instead of strictly real components. The span of all such vectors is denoted by $\mathbb{C}^n$.

Many of the properties of real vectors also apply to complex vectors, but there are exceptions.

For real vectors for instance we have $\|\underline{v}\| = \sqrt{\underline{v} \cdot \underline{v}}$. But now suppose $\underline{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$ so

$$\|\underline{v}\| = \sqrt{\underline{v} \cdot \underline{v}} = \sqrt{i^2 + 1^2} = \sqrt{-1+1} = 0$$

We don't want a non-zero vector to have a norm of 0.

Instead, we define the complex dot product of vectors

$$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \quad \text{and} \quad \underline{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

For complex numbers $v_1, v_2, \dots, v_n$ and $w_1, w_2, \dots, w_n$ to be

$$\underline{v} \cdot \underline{w} = \overline{v_1} w_1 + \overline{v_2} w_2 + \dots + \overline{v_n} w_n$$

Now $\|\underline{v}\| = \sqrt{\underline{v} \cdot \underline{v}} = \sqrt{\overline{i} \cdot i + \overline{1} \cdot 1} = \sqrt{(-i)i + 1 \cdot 1}$

$$= \sqrt{1+1}$$

$$= \sqrt{2}$$