Section 3.2: The Properties of Determinants

It is particularly easy to compute the determinant of a triangular matrix.

eg

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ -5 & 7 & 0 & 0 & 0 \\ 1 & -1 & -3 & 0 & 0 \\ 0 & 4 & 5 & -1 & 0 \\ 9 & 8 & 7 & 6 & 5 \end{bmatrix}$$

We expand along the 1st row:

$$\det(A) = 2 \begin{vmatrix} 7 & 0 & 0 & 0 \\ -1 & -3 & 0 & 0 \\ 4 & 5 & -1 & 0 \\ 8 & 7 & 6 & 5 \end{vmatrix} - 0 + 0 - 0 + 0$$

$$= 2 \cdot 7 \begin{vmatrix} -3 & 0 & 0 \\ 5 & -1 & 0 \\ 7 & 6 & 5 \end{vmatrix} - 0 + 0 - 0$$

$$= 2 \cdot 7 \cdot (-3) \begin{vmatrix} -1 & 0 \\ 6 & 5 \end{vmatrix} - 0 + 0$$

$$= 2 \cdot 7 \cdot (-3) \cdot (-1) \cdot 5$$

$$\boxed{= 210}$$

Theorem: The determinant of a square triangular matrix is the product of the entries on its main diagonal.

Def'n: A <u>linear function</u> from a set $\mathcal{D}$ to a set $\mathcal{R}$ is a function f satisfying both

① $f(\underline{x}+\underline{y}) = f(\underline{x}) + f(\underline{y})$ for all elements $\underline{x}$ and $\underline{y}$ in $\mathcal{D}$, and

② $f(k\underline{x}) = k f(\underline{x})$ for all scalars k and all elements $\underline{x}$ in $\mathcal{D}$.

eg The function $f(x) = 5x$ from $\mathbb{R}$ to $\mathbb{R}$ is linear because

$$f(x+y) = 5(x+y) = 5x+5y = f(x)+f(y)$$

and $f(kx) = 5(kx) = k(5x) = kf(x)$

eg The function $f(x) = 5x+2$ from $\mathbb{R}$ to $\mathbb{R}$ is not linear in this sense because

$$f(x+y) = 5(x+y)+2 = 5x+5y+2$$

but $f(x)+f(y) = (5x+2)+(5y+2) = 5x+5y+4$

The determinant is a function which maps $n \times n$ matrices to real numbers $\mathbb{R}$, but it is <u>not</u> linear.

eg If $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}$ then

$\det(A) = 4-1 = 3$ and $\det(B) = 4-1 = 3$

However, $A+B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ so $\det(A+B) = 0$
$\neq \det(A)+\det(B)$

But consider an $n \times n$ matrix $A = \begin{bmatrix} \underline{a}_1 \rightarrow \\ \underline{a}_2 \rightarrow \\ \vdots \\ \underline{a}_n \rightarrow \end{bmatrix}$. We will denote

$$\det(A) = \det(\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n).$$

__Theorem__: If A is a square matrix with rows $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n$ and $\underline{b}$ is a vector then

① $\det(\underline{a}_1, \ldots, \underline{a}_i + \underline{b}, \ldots, \underline{a}_n)$
$= \det(\underline{a}_1, \ldots, \underline{a}_i, \ldots, \underline{a}_n) + \det(\underline{a}_1, \ldots, \underline{b}, \ldots, \underline{a}_n)$

② $\det(\underline{a}_1, \ldots, k\underline{a}_i, \ldots, \underline{a}_n)$
$= k \det(\underline{a}_1, \ldots, \underline{a}_i, \ldots, \underline{a}_n)$ for any scalar k.

We say that the determinant is __linear in each row__.

eg Given $A = \begin{bmatrix} a & b \\ c+x & d+y \end{bmatrix}$ then $\det(A) = \begin{vmatrix} a & b \\ c+x & d+y \end{vmatrix}$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ x & y \end{vmatrix}$$

Theorem: If A is an $n \times n$ matrix and k is any scalar then $\det(kA) = k^n \det(A)$.

Theorem: If a matrix has a row of zeros then its determinant is zero.

Theorem: The determinant of a matrix changes sign if two rows are interchanged.

eg Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ so $\det(A) = ad - bc$.

Then $B = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$ so

$$\begin{aligned} \det(B) &= bc - ad \\ &= -(ad - bc) \\ &= -\det(A). \end{aligned}$$

Theorem: If a square matrix A has two equal rows then $\det(A) = 0$.

Proof: Assume that A has rows $\underline{a}_1, \dots, \underline{a}_n$ where, for some i and j, $\underline{a}_i = \underline{a}_j = \underline{b}$.

Then
$$\det(A) = \det(\underline{a}_1, \dots, \underline{a}_i, \dots, \underline{a}_j, \dots, \underline{a}_n)$$
$$= -\det(\underline{a}_1, \dots, \underline{a}_j, \dots, \underline{a}_i, \dots, \underline{a}_n)$$
$$= -\det(\underline{a}_1, \dots, \underline{b}, \dots, \underline{b}, \dots, \underline{a}_n)$$

But this means that
$$\det(\underline{a}_1, \dots, \underline{b}, \dots, \underline{b}, \dots, \underline{a}_n) = -\det(\underline{a}_1, \dots, \underline{b}, \dots, \underline{b}, \dots \underline{a}_n)$$
$$2\det(\underline{a}_1, \dots, \underline{b}, \dots, \underline{b}, \dots, \underline{a}_n) = 0$$
$$\boxed{\det(A) = 0}$$

Theorem: If a square matrix A has one row which is a scalar multiple of another row then $\det(A) = 0$.

Proof: Assume that A has rows $\underline{a}_1, \dots, \underline{a}_n$ where $\underline{a}_i = k\underline{a}_j$ for some constant k. Then

$$\det(A) = \det(\underline{a}_1, \dots, \underline{a}_i, \dots, \underline{a}_j, \dots, \underline{a}_n)$$
$$= \det(\underline{a}_1, \dots, k\underline{a}_j, \dots, \underline{a}_j, \dots, \underline{a}_n)$$
$$= k\det(\underline{a}_1, \dots, \underline{a}_j, \dots, \underline{a}_j, \dots, \underline{a}_n)$$
$= k \cdot 0$ because the determinant now includes two equal rows

$\boxed{= 0}$

Theorem : The determinant of a square matrix is unchanged when a multiple of one of its rows is subtracted from another of its rows.

Proof : If A has rows $\underline{a}_1, \dots, \underline{a}_n$ then

$$\det(A) = \det(\underline{a}_1, \dots, \underline{a}_n).$$

Now consider a matrix B which is obtained by replacing $\underline{a}_i$ with $\underline{a}_i - k\underline{a}_j$ for some constant k.

Then
$$\begin{aligned}
\det(B) &= \det(\underline{a}_1, \dots, \underline{a}_i - k\underline{a}_j, \dots, \underline{a}_j, \dots, \underline{a}_n) \\
&= \det(\underline{a}_1, \dots, \underline{a}_i, \dots, \underline{a}_j, \dots, \underline{a}_n) \\
&\quad - \det(\underline{a}_1, \dots, k\underline{a}_j, \dots, \underline{a}_j, \dots, \underline{a}_n) \\
&= \det(A) - 0 \quad \text{because row } i \text{ is a constant multiple of row } j \\
&\boxed{= \det(A)}
\end{aligned}$$

eg Find the determinant of $A = \begin{bmatrix} 2 & 2 & -2 & -1 \\ -1 & 4 & 1 & -5 \\ 1 & -6 & -4 & 9 \\ -3 & 8 & 7 & 1 \end{bmatrix}$

Instead of using the Laplace expansion directly we will first bring A to row-echelon form.

$$\begin{bmatrix} 2 & 2 & -2 & -1 \\ -1 & 4 & 1 & -5 \\ 1 & -6 & -4 & 9 \\ -3 & 8 & 7 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & -6 & -4 & 9 \\ -1 & 4 & 1 & -5 \\ 2 & 2 & -2 & -1 \\ -3 & 8 & 7 & 1 \end{bmatrix}$$

$$\xrightarrow[R_4 \to R_4 - (-3)R_1]{R_2 \to R_2 - (-1)R_1,\; R_3 \to R_3 - 2R_1} \begin{bmatrix} 1 & -6 & -4 & 9 \\ 0 & -2 & -3 & 4 \\ 0 & 14 & 6 & -19 \\ 0 & -10 & -5 & 28 \end{bmatrix}$$

$$\xrightarrow{R_2 \to -\frac{1}{2}R_2} \begin{bmatrix} 1 & -6 & -4 & 9 \\ 0 & 1 & 3/2 & -2 \\ 0 & 14 & 6 & -19 \\ 0 & -10 & -5 & 28 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 14R_2,\; R_4 \to R_4 - (-10)R_2} \begin{bmatrix} 1 & -6 & -4 & 9 \\ 0 & 1 & 3/2 & -2 \\ 0 & 0 & -15 & 9 \\ 0 & 0 & 10 & 8 \end{bmatrix}$$

$$\xrightarrow{R_3 \to -\frac{1}{15}R_3} \begin{bmatrix} 1 & -6 & -4 & 9 \\ 0 & 1 & 3/2 & -2 \\ 0 & 0 & 1 & -3/5 \\ 0 & 0 & 10 & 8 \end{bmatrix}$$

$$\xrightarrow{R_4 \to R_4 - 10R_3} \begin{bmatrix} 1 & -6 & -4 & 9 \\ 0 & 1 & 3/2 & -2 \\ 0 & 0 & 1 & -3/5 \\ 0 & 0 & 0 & 14 \end{bmatrix} = U$$

Immediately $\det(U) = 1 \cdot 1 \cdot 1 \cdot 14 = 14$

But we also have that $\det(U) = (-1)(-\frac{1}{2})(-\frac{1}{15})\det(A)$

$$14 = -\frac{1}{30}\det(A)$$

$$\boxed{\det(A) = -420}$$

Theorem: The determinant is multiplicative: $\det(AB) = \det(A)\det(B)$.

Theorem: If A is invertible then $\det(A) \neq 0$ and

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Proof: There exists a matrix A^{-1} for which $AA^{-1} = I$ so

$$\det(AA^{-1}) = \det(I)$$
$$\det(A)\det(A^{-1}) = \det(I)$$
$$\det(A)\det(A^{-1}) = 1$$

because I is a diagonal matrix with only 1 along the diagonal

Thus it must be that $\det(A) \neq 0$ and $\det(A^{-1}) \neq 0$.

Furthermore we can write

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

eg If $A = \begin{bmatrix} 3 & 7 & -3 & 1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$ then A must be invertible since $\det(A) = 3 \cdot 1 \cdot 5 \cdot (-2)$
$= -30 \neq 0$.

<u>Def'n</u>: A square matrix is <u>singular</u> if it is not invertible.
A singular matrix has a zero determinant.

eg Find all values of k which would make A singular given

$$A = \begin{bmatrix} k & 1 & 0 \\ 0 & 2 & k \\ -1 & k & 1 \end{bmatrix}.$$

We want to know when $\det(A) = 0$ so we compute

$$\det(A) = k\begin{vmatrix} 2 & k \\ k & 1 \end{vmatrix} - \begin{vmatrix} 0 & k \\ -1 & 1 \end{vmatrix} + 0$$

$$= k(2-k^2) - (0+k)$$

$$= 2k - k^3 - k$$

$$= k - k^3$$

We set $k - k^3 = 0$

$$-k(k^2-1) = 0$$

$$-k(k-1)(k+1) = 0 \quad \text{so } \boxed{k=0,\ k=1,\ k=-1}$$

<u>Theorem</u>: For any square matrix A, $\det(A) = \det(A^T)$.

<u>Proof</u>: If C is the matrix of cofactors of A then

$$AC^T = C^TA = (\det A)\,I$$

But then

$$(AC^T)^T = (C^TA)^T = [(\det A)\,I]^T$$

$$(C^T)^T A^T = A^T (C^T)^T = (\det A) I$$
$$C A^T = A^T C = (\det A) I$$

Likewise, C^T is the matrix of cofactors of A^T, so

$$A^T (C^T)^T = (C^T)^T A^T = (\det A^T) I$$
$$A^T C = C A^T = (\det A^T) I$$

Thus $(\det A) I = (\det A^T) I$ so $\det(A) = \det(A^T)$.