Section 2.5: Finding the Inverse of a Matrix

Given any square matrix A, we know that B is its inverse if $AB = I$. So if $B = \begin{bmatrix} \underline{x}_1 & \underline{x}_2 & \cdots & \underline{x}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$ then

$$AB = \begin{bmatrix} A\underline{x}_1 & A\underline{x}_2 & \cdots & A\underline{x}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} = I = \begin{bmatrix} \underline{e}_1 & \underline{e}_2 & \cdots & \underline{e}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

Thus we must have $A\underline{x}_1 = \underline{e}_1$, $A\underline{x}_2 = \underline{e}_2, \ldots,$ $A\underline{x}_n = \underline{e}_n$.

eg Find the inverse of $A = \begin{bmatrix} 2 & 9 \\ 1 & 3 \end{bmatrix}$.

If $B = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$ then we must have

$$\begin{bmatrix} 2 & 9 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This gives us 2 systems of equations:

$$\begin{bmatrix} 2 & 9 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and $$\begin{bmatrix} 2 & 9 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Then $\left[\begin{array}{cc|c} 2 & 9 & 1 \\ 1 & 3 & 0 \end{array}\right]$ $\qquad$ $\left[\begin{array}{cc|c} 2 & 9 & 0 \\ 1 & 3 & 1 \end{array}\right]$

$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & 9 & 1 \end{array}\right]$ $\qquad$ $\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 2 & 9 & 0 \end{array}\right]$

$\xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 3 & 1 \end{array}\right]$ $\qquad$ $\xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 3 & -2 \end{array}\right]$

$\xrightarrow{R_2 \to \frac{1}{3}R_2} \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 1 & 1/3 \end{array}\right]$ $\qquad$ $\xrightarrow{R_2 \to \frac{1}{3}R_2} \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 1 & -2/3 \end{array}\right]$

Both systems are solved through identical row operations, and even back-substitution can be aligned if we make the entry <u>above</u> the pivot in the 2nd column a zero.

$\xrightarrow{R_1 \to R_1 - 3R_2} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1/3 \end{array}\right]$ $\qquad$ $\xrightarrow{R_1 \to R_1 - 3R_2} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2/3 \end{array}\right]$

Thus $x_1 = -1$ $\qquad$ $y_1 = 3$

$x_2 = 1/3$ $\qquad$ $y_2 = -2/3$

And so $A^{-1} = \begin{bmatrix} -1 & 3 \\ 1/3 & -2/3 \end{bmatrix}$

Compare: $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \dfrac{1}{-3}\begin{bmatrix} 3 & -9 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1/3 & -2/3 \end{bmatrix}$

Thus we have shown that $\left[\begin{array}{cc|cc} 2 & 9 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right]$ can be row-reduced to $\left[\begin{array}{cc|cc} 1 & 0 & -1 & 3 \\ 0 & 1 & 1/3 & -2/3 \end{array}\right]$ where the matrix on the right is A^{-1}.

So for any matrix A, $[A \mid I]$ being row-reduced to $[I \mid B]$ implies that A is invertible and $B = A^{-1}$. If this is not possible then A is not invertible.

<u>Def'n</u>: A matrix is in <u>reduced row-echelon form</u> if it is in row-echelon form, each pivot is 1, and each pivot is the only non-zero entry in its column.

eg $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$ is in reduced row-echelon form

eg $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$ is in row-echelon form but not reduced row-echelon form, because there is a non-zero entry above the pivot in Column 3 eg Find the inverse of $A = \begin{bmatrix} 1 & -1 & 2 \\ -5 & 7 & -11 \\ -2 & 3 & -5 \end{bmatrix}$ if possible.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ -5 & 7 & -11 & 0 & 1 & 0 \\ -2 & 3 & -5 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - (-2)R_1]{R_2 \to R_2 - (-5)R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & 5 & 1 & 0 \\ 0 & 1 & -1 & 2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & 0 & 1 \\ 0 & 2 & -1 & 5 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - 2R_2]{R_1 \to R_1 - (-1)R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & 0 & 1 \\ 0 & 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & -2 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 - (-1)R_3]{R_1 \to R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 3 \\ 0 & 1 & 0 & 3 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 & -2 \end{array}\right]$$

Thus A is invertible and $A^{-1} = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & -1 \\ 1 & 1 & -2 \end{bmatrix}$

eg Find the inverse of $A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ if possible.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - (-1)R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

Because we have obtained a row of zeros A cannot be brought to I.

Thus A is <u>not</u> invertible.

eg Solve the system

$$\begin{cases} x_1 + 4x_2 + 2x_3 = 1 \\ 2x_1 + 3x_2 + 3x_3 = -1 \\ 4x_1 + x_2 + 4x_3 = 0 \end{cases}$$

Instead of back-substitution, we could now use the fact that if $A\underline{x} = \underline{b}$ then $\underline{x} = A^{-1}\underline{b}$ for an invertible matrix A.

Here we have

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 2 & 3 & 3 & 0 & 1 & 0 \\ 4 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & -5 & -1 & -2 & 1 & 0 \\ 0 & -15 & -4 & -4 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to (-\frac{1}{5})R_2} \left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1/5 & 2/5 & -1/5 & 0 \\ 0 & -15 & -4 & -4 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - (-15)R_1]{R_1 \to R_1 - 4R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 6/5 & -3/5 & 4/5 & 0 \\ 0 & 1 & 1/5 & 2/5 & -1/5 & 0 \\ 0 & 0 & -1 & 2 & -3 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to (-1)R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 6/5 & -3/5 & 4/5 & 0 \\ 0 & 1 & 1/5 & 2/5 & -1/5 & 0 \\ 0 & 0 & 1 & -2 & 3 & -1 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 - \frac{1}{5}R_3]{R_1 \to R_1 - \frac{6}{5}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9/5 & -14/5 & 6/5 \\ 0 & 1 & 0 & 4/5 & -4/5 & 1/5 \\ 0 & 0 & 1 & -2 & 3 & -1 \end{array}\right]$$

Thus $A^{-1} = \begin{bmatrix} 9/5 & -14/5 & 6/5 \\ 4/5 & -4/5 & 1/5 \\ -2 & 3 & -1 \end{bmatrix}$ and so

$$\underline{x} = \begin{bmatrix} 9/5 & -14/5 & 6/5 \\ 4/5 & -4/5 & 1/5 \\ -2 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 23/5 \\ 8/5 \\ -5 \end{bmatrix}.$$

Def'n: An elementary matrix is a square matrix obtained from the identity matrix by a single elementary row operation.

eg

$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ can be obtained from I by interchanging R_2 and R_4

$E_2 = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$ can be obtained from I by replacing R_1 with $5R_1$

$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ can be obtained from I by replacing R_2 with $R_2 - 2R_1$

Any row operation on a matrix A is equivalent to left-multiplying A by the corresponding elementary matrix.

eg If $A = \begin{bmatrix} 2 & 1 \\ 0 & -4 \\ 6 & 6 \\ -1 & 3 \end{bmatrix}$ then, using E_1 from above,

$$E_1 A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & -4 \\ 6 & 6 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 6 & 6 \\ 0 & -4 \end{bmatrix}$$

So we have interchanged R_2 and R_4.

All elementary matrices are invertible, and E^{-1} represents the row operation which undoes the operation corresponding to E.

eg $E_2 = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$ has inverse $E_2^{-1} = \begin{bmatrix} 1/5 & 0 \\ 0 & 1 \end{bmatrix}$

<u>Theorem</u>: A matrix A is invertible if and only if A can be written as the product of elementary matrices.

<u>Proof</u>: First assume A is invertible. We must prove that it can be written as the product of elementary matrices.

Since A is invertible, a sequence of row operations will transform it to I. If $E_1, E_2, \ldots, E_n$ are the corresponding elementary matrices then

$$E_n \cdots E_2 E_1 A = I$$
$$A^{-1} = E_n \cdots E_2 E_1$$
$$A = (E_n \cdots E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} \cdots E_n^{-1}$$

Thus A is the product of elementary matrices E_1^{-1}, $E_2^{-1}, \ldots, E_n^{-1}$.

Now we assume that A is the product of elementary matrices. We must prove that A is invertible.

Then $A = E_1 E_2 \cdots E_n$. But all elementary matrices are invertible, and the product of invertible matrices is invertible, so A is invertible.

eg Express $A = \begin{bmatrix} 2 & 9 \\ 1 & 3 \end{bmatrix}$ as the product of elementary matrices.

We have seen that A can be reduced to I as its reduced row echelon form. The corresponding elementary matrices are

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix},$$

$$E_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

Then $E_4 E_3 E_2 E_1 A = I$

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

Theorem: A square matrix A is invertible if and only if the columns of A are linearly independent.

Proof: First we assume that A is invertible. Then consider the equation $A\underline{x} = \underline{0}$. We have

$$A^{-1}A\underline{x} = A^{-1}\underline{0}$$

$$\underline{x} = \underline{0}$$

So there is only the trivial solution, and hence the columns of A are linearly independent.

Now suppose A has linearly independent columns. Then $A\underline{x} = \underline{0}$ has only the trivial solution $\underline{x} = \underline{0}$. But if the equation has a unique solution then there are no free variables, so all the columns of A are pivot columns. Then the reduced row-echelon form of A must be I, so A is the product of elementary matrices and is therefore invertible.