Section 2.4: Homogeneous Systems and Linear Independence

A system of equations is homogeneous if it can be written in the form $A\underline{x} = \underline{0}$.

eg $\begin{cases} 4x_1 + x_2 - 2x_3 = 0 \\ x_1 - 2x_2 + x_3 = 0 \end{cases}$ is homogeneous

Given a system $A\underline{x} = \underline{b}$ then the corresponding homogeneous system is given by $A\underline{x} = \underline{0}$ for the same matrix A.

eg The previous homogeneous system is the homogeneous system corresponding to

$$\begin{cases} 4x_1 + x_2 - 2x_3 = 0 \\ x_1 - 2x_2 + x_3 = 4 \end{cases}$$

Since $\underline{x} = \underline{0}$ is always a solution of a homogeneous system it is called the trivial solution. Thus the only question is whether there are also non-trivial solutions, in which case there is infinitely many of them.

When applying Gaussian elimination to a homogeneous system we do not consider an augmented matrix because the righthand side will always consist only of zeros.

eg $\begin{cases} 4x_1 + x_2 - 2x_3 = 0 \\ x_1 - 2x_2 + x_3 = 0 \end{cases}$

$$\begin{bmatrix} 4 & 1 & -2 \\ 1 & -2 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -2 & 1 \\ 4 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - 4R_1} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 9 & -6 \end{bmatrix}$$

$$\xrightarrow{R_2 \to \frac{1}{9}R_2} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2/3 \end{bmatrix}$$

Then $x_3 = t$

$x_2 - \frac{2}{3}x_3 = 0 \to x_2 = \frac{2}{3}t$

$x_1 - 2x_2 + x_3 = 0 \to x_1 = 2\left(\frac{2}{3}t\right) - t = \frac{1}{3}t$

We have infinitely many non-trivial solutions of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}t \\ \frac{2}{3}t \\ t \end{bmatrix} = t \begin{bmatrix} 1/3 \\ 2/3 \\ 1 \end{bmatrix}$$

Recall that in Section 2.3 we showed that the solutions of the non-homogeneous system

$\begin{cases} 4x_1 + x_2 - 2x_3 = 0 \\ x_1 - 2x_2 + x_3 = 4 \end{cases}$

were $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4/9 \\ -16/9 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1/3 \\ 2/3 \\ 1 \end{bmatrix}$

Theorem: Suppose that $\underline{x}_p$ is a particular solution to the non-homogeneous system $A\underline{x} = \underline{b}$ and $\underline{y}$ is any other solution. Then $\underline{y} = \underline{x}_p + \underline{x}_h$ where $\underline{x}_h$ is a solution to the corresponding homogeneous system $A\underline{x} = \underline{0}$.

Proof: We have $A\underline{x}_p = \underline{b}$ and $A\underline{y} = \underline{b}$. Then

$$A\underline{y} - A\underline{x}_p = \underline{b} - \underline{b}$$

$$A\underline{y} - A\underline{x}_p = \underline{0}$$

$$A(\underline{y} - \underline{x}_p) = \underline{0}$$

Hence $\underline{y} - \underline{x}_p$ is a solution to the corresponding homogeneous system $A\underline{x} = \underline{0}$, and so we can write

$$\underline{y} - \underline{x}_p = \underline{x}_h$$

$$\underline{y} = \underline{x}_p + \underline{x}_h.$$

Recall that vectors $\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n$ are linearly independent if the equation

$$k_1\underline{x}_1 + k_2\underline{x}_2 + \dots + k_n\underline{x}_n = \underline{0}$$

has only the trivial solution $k_1 = k_2 = \dots = k_n = 0$. Otherwise, they are linearly dependent.

We can write this as the matrix equation $A\underline{k} = \underline{0}$ where

$$A = \begin{bmatrix} \underline{x}_1 & \underline{x}_2 & \cdots & \underline{x}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \quad \text{and} \quad \underline{k} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}.$$

If this homogeneous system has only a trivial solution then these vectors are linearly independent. Otherwise they are linearly dependent.

Thus the vectors are linearly independent only if all the columns of A are pivot columns.

eg Determine whether $\underline{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}$, $\underline{x}_2 = \begin{bmatrix} -5 \\ -3 \\ 5 \\ 1 \end{bmatrix}$,

$\underline{x}_3 = \begin{bmatrix} 14 \\ 14 \\ -2 \\ -2 \end{bmatrix}$ are linearly independent or linearly dependent.

We set $k_1 \underline{x}_1 + k_2 \underline{x}_2 + k_3 \underline{x}_3 = \underline{0}$ so

$$\begin{cases} k_1 - 5k_2 + 14k_3 = 0 \\ 2k_1 - 3k_2 + 14k_3 = 0 \\ 2k_1 + 5k_2 - 2k_3 = 0 \\ k_2 - 2k_3 = 0 \end{cases}$$

Thus

$$A = \begin{bmatrix} 1 & -5 & 14 \\ 2 & -3 & 14 \\ 2 & 5 & -2 \\ 0 & 1 & -2 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & -5 & 14 \\ 0 & 7 & -14 \\ 0 & 15 & -30 \\ 0 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow{R_2 \to \frac{1}{7}R_2} \begin{bmatrix} 1 & -5 & 14 \\ 0 & 1 & -2 \\ 0 & 15 & -30 \\ 0 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow[R_4 \to R_4 - R_2]{R_3 \to R_3 - 15R_2} \begin{bmatrix} 1 & -5 & 14 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Here Column 3 is not a pivot column, so there are infinitely many solutions. Thus these vectors are <u>linearly dependent</u>.

To represent one vector as a linear combination of the others we can set

$$k_3 = t$$
$$k_2 = 2k_3 = 2t$$
$$k_1 = 5k_2 - 14k_3 = 5(2t) - 14t = -4t$$

So for $t=1$, we would have $k_1 = -4$, $k_2 = 2$, $k_3 = 1$ and then

$$-4\underline{x}_1 + 2\underline{x}_2 + \underline{x}_3 = \underline{0}$$
$$\underline{x}_3 = 4\underline{x}_1 - 2\underline{x}_2 .$$