Section 2.3: Systems of Linear Equations

Suppose we have a system of m equations in n unknowns, usually denoted by $x_1, x_2, x_3, \dots, x_n$.

$$\text{eg} \begin{cases} -3x_1 - 3x_2 + x_3 = -8 \\ 5x_1 \quad\quad - 2x_3 = 20 \\ x_1 - 4x_2 + 3x_3 = -9 \end{cases}$$

One thing we can do is interchange any 2 equations.

eg If we swap Equation ① and Equation ③ we get

$$\begin{cases} x_1 - 4x_2 + 3x_3 = -9 \\ 5x_1 \quad\quad - 2x_3 = 20 \\ -3x_1 - 3x_2 + x_3 = -8 \end{cases}$$

We can also subtract multiples of one equation from another equation.

eg If we subtract 5 times Equation ① from Equation ② and -3 times Equation ① from Equation ③ we get

$$\begin{cases} x_1 - 4x_2 + 3x_3 = -9 \\ 20x_2 - 17x_3 = 65 \\ -15x_2 + 10x_3 = -35 \end{cases}$$

We can also multiply any equation by a non-zero constant.

eg If we multiply Equation ② by $\frac{1}{20}$ we have

$$\begin{cases} x_1 - 4x_2 + 3x_3 = -9 \\ x_2 - \frac{17}{20}x_3 = \frac{13}{4} \\ -15x_2 + 10x_3 = -35 \end{cases}$$

Now we subtract -15 times Equation ② from Equation ③ to get

$$\begin{cases} x_1 - 4x_2 + 3x_3 = -9 \\ x_2 - \frac{17}{20}x_3 = \frac{13}{4} \\ -\frac{11}{4}x_3 = \frac{55}{4} \end{cases}$$

Finally we multiply Equation ③ by $-\frac{4}{11}$ to get

$$\begin{cases} x_1 - 4x_2 + 3x_3 = -9 \\ x_2 - \frac{17}{20}x_3 = \frac{13}{4} \\ x_3 = -5 \end{cases}$$

Now we can use <u>back-substitution</u> to find x_2 and then x_1:

If $x_3 = -5$, $x_2 - \frac{17}{20} \cdot (-5) = \frac{13}{4} \rightarrow x_2 = -1$

$x_1 - 4(-1) + 3(-5) = -9 \rightarrow x_1 = 2$

So the solution of the system is $\boxed{x_1 = 2,\ x_2 = -1,\ x_3 = -5}$

We have already observed that this system can be written as a matrix equation:

$$\begin{bmatrix} -3 & -3 & 1 \\ 5 & 0 & -2 \\ 1 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 20 \\ -9 \end{bmatrix}$$

But this can also be represented using an <u>augmented matrix</u>

$$\left[\begin{array}{ccc|c} -3 & -3 & 1 & -8 \\ 5 & 0 & -2 & 20 \\ 1 & -4 & 3 & -9 \end{array}\right] = [A \mid \underline{b}]$$

Each operation that can be performed on the system of equations is equivalent to a <u>row operation</u> on the augmented matrix:

1. Two rows can be interchanged $\quad R_i \leftrightarrow R_j$
2. Any row can be multiplied by a non-zero scalar k, called a <u>multiplier</u> $\quad R_i \rightarrow kR_i$
3. Any row can be replaced by that row minus a multiple of another row $\quad R_i \rightarrow R_i - kR_j$

eg

$$\left[\begin{array}{ccc|c} -3 & -3 & 1 & -8 \\ 5 & 0 & -2 & 20 \\ 1 & -4 & 3 & -9 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -4 & 3 & -9 \\ 5 & 0 & -2 & 20 \\ -3 & -3 & 1 & -8 \end{array}\right]$$

$$\xrightarrow[R_3 \rightarrow R_3 - (-3)R_1]{R_2 \rightarrow R_2 - 5R_1} \left[\begin{array}{ccc|c} 1 & -4 & 3 & -9 \\ 0 & 20 & -17 & 65 \\ 0 & -15 & 10 & -35 \end{array}\right]$$

$$\xrightarrow{R_2 \to \frac{1}{20}R_2} \left[\begin{array}{ccc|c} 1 & -4 & 3 & -9 \\ 0 & 1 & -17/20 & 13/4 \\ 0 & -15 & 10 & -35 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3-(-15)R_2} \left[\begin{array}{ccc|c} 1 & -4 & 3 & -9 \\ 0 & 1 & -17/20 & 13/4 \\ 0 & 0 & -11/4 & 55/4 \end{array}\right]$$

$$\xrightarrow{R_3 \to (-\frac{4}{11})R_3} \left[\begin{array}{ccc|c} 1 & -4 & 3 & -9 \\ 0 & 1 & -17/20 & 13/4 \\ 0 & 0 & 1 & -5 \end{array}\right]$$

This is <u>Gaussian elimination</u>.

1. Interchange rows (if necessary) so that there are no non-zero entries to the left of the first non-zero entry in R_1. This first non-zero entry in the row is called a <u>pivot</u>, and the column in which it appears is a <u>pivot column</u>.

2. Divide R_1 by the pivot.

3. Subtract multiples of R_1 from the other rows so that all the entries below the pivot are zero.

4. Repeat steps 1 to 3 on $R_2, R_3, \ldots$ until we reach the last non-zero row. Then we use back-substitution to solve the system.

eg Solve $\begin{cases} x_1 - x_2 - x_3 = -10 \\ x_1 + 5x_2 + x_3 = 2 \\ 3x_1 \quad\quad + 2x_3 = 0 \end{cases}$

This can be written

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & -10 \\ 1 & 5 & 1 & 2 \\ 3 & 0 & 2 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & -1 & -1 & -10 \\ 0 & 6 & 2 & 12 \\ 0 & 3 & 5 & 30 \end{array}\right]$$

$$\xrightarrow{R_2 \to \frac{1}{6}R_2} \left[\begin{array}{ccc|c} 1 & -1 & -1 & -10 \\ 0 & 1 & 1/3 & 2 \\ 0 & 3 & 5 & 30 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 3R_2} \left[\begin{array}{ccc|c} 1 & -1 & -1 & -10 \\ 0 & 1 & 1/3 & 2 \\ 0 & 0 & 4 & 24 \end{array}\right]$$

$$\xrightarrow{R_3 \to \frac{1}{4}R_3} \left[\begin{array}{ccc|c} 1 & -1 & -1 & -10 \\ 0 & 1 & 1/3 & 2 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

Now $x_3 = 6$

$x_2 + \frac{1}{3}x_3 = 2 \longrightarrow x_2 + \frac{1}{3} \cdot 6 = 2 \rightarrow x_2 = 0$

$x_1 - x_2 - x_3 = -10 \rightarrow x_1 - 0 - 6 = -10 \rightarrow x_1 = -4$

Through Gaussian elimination we transform A into a special upper triangular matrix called a <u>row-echelon matrix</u>.

Def'n: A row-echelon matrix is an upper triangular matrix with the following properties:

1. All rows consisting entirely of zeros are at the bottom.

2. The pivots step from left to right as the matrix is read from top to bottom.

3. All entries in a column below a pivot are zero.

eg $\begin{bmatrix} 2 & 0 & -3 & 1 \\ 0 & 0 & -5 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is a row-echelon matrix where the pivot columns are columns 1, 3

$\begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & 6 \end{bmatrix}$ is a row-echelon matrix where the pivot columns are columns 2, 3

$\begin{bmatrix} 1 & 4 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ is <u>not</u> a row-echelon matrix because there is a non-zero entry below a pivot

$\begin{bmatrix} -2 & 7 & 1 \\ 0 & 0 & 4 \\ 0 & -8 & -3 \end{bmatrix}$ is <u>not</u> a row-echelon matrix because the pivots do not step from left to right

Some systems have <u>unique</u> solutions, but this is not always the case.

eg Solve $\begin{cases} x-3y=2 \\ 3x-9y=4 \end{cases}$

We have $\left[\begin{array}{cc|c} 1 & -3 & 2 \\ 3 & -9 & 4 \end{array}\right] \xrightarrow{R_2 \to R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & -3 & 2 \\ 0 & 0 & -2 \end{array}\right]$

But this means that $0x + 0y = -2$ which is impossible. Hence there are <u>no solutions</u>, and this system is <u>inconsistent</u>.

<u>Def'n</u>: A system of equations with at least one solution is <u>consistent</u>.

eg Solve $\begin{cases} 4x_1 + x_2 - 2x_3 = 0 \\ x_1 - 2x_2 + x_3 = 4 \end{cases}$

We have $\left[\begin{array}{ccc|c} 4 & 1 & -2 & 0 \\ 1 & -2 & 1 & 4 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 4 & 1 & -2 & 0 \end{array}\right]$

$\xrightarrow{R_2 \to R_2 - 4R_1} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 9 & -6 & -16 \end{array}\right]$

$\xrightarrow{R_2 \to \frac{1}{9}R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 1 & -2/3 & -16/9 \end{array}\right]$

Now we see that $x_2 - \frac{2}{3}x_3 = -\frac{16}{9}$

$$x_1 - 2x_2 + x_3 = 4$$

But although we can write $x_2 = \frac{2}{3}x_3 - \frac{16}{9}$

and $x_1 = 2x_2 - x_3 + 4$

we have no information to specify x_3.

Thus we can assign to x_3 a parameter t, which can represent any real number. Then

$$x_2 = \frac{2}{3}t - \frac{16}{9}$$

$$x_1 = 2\left(\frac{2}{3}t - \frac{16}{9}\right) - t + 4 = \frac{1}{3}t + \frac{4}{9}$$

Hence there is an infinite number of solutions to the system.

When this happens, we usually express the solutions as a vector:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}t + \frac{4}{9} \\ \frac{2}{3}t - \frac{16}{9} \\ t \end{bmatrix} = \begin{bmatrix} \frac{4}{9} \\ -\frac{16}{9} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{3}t \\ \frac{2}{3}t \\ t \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{9} \\ -\frac{16}{9} \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix}$$

This is the general solution of the system, and the parameter(s) are also called free variables.

If we chose a particular value of the free variables, we obtain a particular solution.

eg If $t=3$, we get the particular solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4/9 \\ -16/9 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 1/3 \\ 2/3 \\ 1 \end{bmatrix} = \begin{bmatrix} 13/9 \\ 2/9 \\ 3 \end{bmatrix}.$$

Theorem: If a system is consistent, any free variables will correspond to the columns that are not pivot columns.

eg $\begin{cases} 3x_1 - 12x_2 + 3x_3 + 9x_4 + 9x_5 = 3 \\ -x_1 + 3x_2 - x_3 - x_4 + 2x_5 = -2 \\ 5x_1 - 18x_2 + 5x_3 + 11x_4 + 7x_5 = 13 \end{cases}$

$$\left[\begin{array}{ccccc|c} 3 & -12 & 3 & 9 & 9 & 3 \\ -1 & 3 & -1 & -1 & 2 & -2 \\ 5 & -18 & 5 & 11 & 7 & 13 \end{array}\right]$$

$$\xrightarrow{R_1 \to \frac{1}{3}R_1} \left[\begin{array}{ccccc|c} 1 & -4 & 1 & 3 & 3 & 1 \\ -1 & 3 & -1 & -1 & 2 & -2 \\ 5 & -18 & 5 & 11 & 7 & 13 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_2 - (-1)R_1} \left[\begin{array}{ccccc|c} 1 & -4 & 1 & 3 & 3 & 1 \\ 0 & -1 & 0 & 2 & 5 & -1 \\ 0 & 2 & 0 & -4 & -8 & 8 \end{array}\right]$$

$$\xrightarrow{R_2 \to (-1)R_2} \left[\begin{array}{ccccc|c} 1 & -4 & 1 & 3 & 3 & 1 \\ 0 & 1 & 0 & -2 & -5 & 1 \\ 0 & 2 & 0 & -4 & -8 & 8 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{ccccc|c} 1 & -4 & 1 & 3 & 3 & 1 \\ 0 & 1 & 0 & -2 & -5 & 1 \\ 0 & 0 & 0 & 0 & 2 & 6 \end{array}\right]$$

$$\xrightarrow{R_3 \to \frac{1}{2}R_3} \left[\begin{array}{ccccc|c} 1 & -4 & 1 & 3 & 3 & 1 \\ 0 & 1 & 0 & -2 & -5 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

Neither column 3 nor column 4 is a pivot column, so we assign $x_3 = t$ and $x_4 = s$. Then

$$x_5 = 3$$

$$x_2 - 2x_4 - 5x_5 = 1 \rightarrow x_2 = 2s + 5 \cdot 3 + 1$$
$$= 2s + 16$$

$$x_1 - 4x_2 + x_3 + 3x_4 + 3x_5 = 1$$

$$\rightarrow x_1 = 4(2s+16) - t - 3s - 3 \cdot 3 + 1$$
$$= 8s + 64 - t - 3s - 9 + 1$$
$$= 5s - t + 56$$

Thus $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5s - t + 56 \\ 2s + 16 \\ t \\ s \\ 3 \end{bmatrix} = \begin{bmatrix} 5s \\ 2s \\ 0 \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} -t \\ 0 \\ t \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 56 \\ 16 \\ 0 \\ 0 \\ 3 \end{bmatrix}$

$$= s \begin{bmatrix} 5 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 56 \\ 16 \\ 0 \\ 0 \\ 3 \end{bmatrix}$$

Here, $s = t = 0$ gives the particular solution $\begin{bmatrix} 56 \\ 16 \\ 0 \\ 0 \\ 3 \end{bmatrix}$

while $s = 1$ and $t = 2$ gives the particular solution $\begin{bmatrix} 59 \\ 18 \\ 2 \\ 1 \\ 3 \end{bmatrix}$

Every system of linear equations has either a unique solution, no solutions, or an infinite number of solutions.

eg Find conditions on a, b, c such that the following system has a unique solution, no solutions, or infinite solutions:

$$\begin{cases} x_1 + x_2 - x_3 = a \\ 2x_1 - 3x_2 + 5x_3 = b \\ 5x_1 \qquad + 2x_3 = c \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & a \\ 2 & -3 & 5 & b \\ 5 & 0 & 2 & c \end{array}\right] \xrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & -1 & a \\ 0 & -5 & 7 & b-2a \\ 0 & -5 & 7 & c-5a \end{array}\right]$$

$$\xrightarrow{R_2 \to -\frac{1}{5}R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & a \\ 0 & 1 & -7/5 & -\frac{1}{5}(b-2a) \\ 0 & -5 & 7 & c-5a \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - (-5)R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & a \\ 0 & 1 & -7/5 & -\frac{1}{5}(b-2a) \\ 0 & 0 & 0 & \underbrace{c-5a-b+2a}_{c-b-3a} \end{array}\right]$$

Column 3 is not a pivot column, so we cannot have a unique solution for any values of a, b, c.

There will be no solutions if $c - b - 3a \neq 0$.

There will be infinitely many solutions if $c - b - 3a = 0$.

Section 2.4: Homogeneous Systems and Linear Independence

A system of equations is homogeneous if it can be written in the form $A\underline{x} = \underline{0}$.

eg $\begin{cases} 4x_1 + x_2 - 2x_3 = 0 \\ x_1 - 2x_2 + x_3 = 0 \end{cases}$ is homogeneous

Given a system $A\underline{x} = \underline{b}$ then the corresponding homogeneous system is given by $A\underline{x} = \underline{0}$ for the same matrix A.

eg The previous homogeneous system is the homogeneous system corresponding to

$$\begin{cases} 4x_1 + x_2 - 2x_3 = 0 \\ x_1 - 2x_2 + x_3 = 4 \end{cases}$$

Since $\underline{x} = \underline{0}$ is always a solution of a homogeneous system it is called the trivial solution. Thus the only question is whether there are also non-trivial solutions, in which case there is infinitely many of them.

When applying Gaussian elimination to a homogeneous system we do not consider an augmented matrix because the righthand side will always consist only of zeros.