Section 2.2: The Inverse and Transpose of a Matrix

__Def'n__: A matrix A has an <u>inverse</u> (or is <u>invertible</u>) if there exists a matrix B for which $AB = I$ and $BA = I$. Then we write $B = A^{-1}$.

eg The inverse of $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ is $A^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$

because $AA^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$A^{-1}A = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

If $B = A^{-1}$ then $A = B^{-1}$. Hence $(A^{-1})^{-1} = A$.

__Theorem__: If a matrix A is invertible then its inverse is unique.

__Proof__: Suppose B and C are both the inverses of A.
Then $AB = BA = I$ and $AC = CA = I$.
But then $BAC = (BA)C = IC = C$

$BAC = B(AC) = BI = B$

Hence $B = C$, so the inverse of A is unique.

Many matrices are not invertible, including but not limited to, the zero matrix $\underline{0}$.

eg Show that $A = \begin{bmatrix} 3 & 1 \\ 9 & 3 \end{bmatrix}$ is not invertible.

If A is invertible then there is a matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

for which $AB = I$

$$\begin{bmatrix} 3 & 1 \\ 9 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3a+c & 3b+d \\ 9a+3c & 9b+3d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

But then $3a+c=1$ and $9a+3c=0$

$$3(3a+c)=0$$

$$3a+c=0$$

These cannot both be true, so there are no solutions to this system. Hence $AB \neq I$ and so no $B = A^{-1}$ exists.

Furthermore it can be shown that only ~~rectangular~~ square matrices (but not all ~~rectangular~~ square matrices) are invertible.

However, if A is square and $AB = I$ then it will always be that $BA = I$ as well.

<u>Theorem</u>: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad-bc \neq 0$ then

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Proof: Here, $AA^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$= \frac{1}{ad-bc} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{ad-bc} \begin{bmatrix} ad-bc & -ab+ab \\ cd-cd & -bc+ad \end{bmatrix}$$

$$= \frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = I$$

Theorem: Given invertible matrices A and B,

$$(AB)^{-1} = B^{-1}A^{-1}$$

Proof: We have $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$

$$= AIA^{-1}$$
$$= AA^{-1}$$
$$= I$$

so $(AB)^{-1} = B^{-1}A^{-1}$.

eg Given $A = \begin{bmatrix} 4 & 0 \\ -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -1 \\ 2 & 3 \end{bmatrix}$,

$$AB = \begin{bmatrix} 4 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ 4 & 5 \end{bmatrix}$$

$$(AB)^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{-4}\begin{bmatrix} 5 & 4 \\ -4 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} -5/4 & -1 \\ 1 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1/4 & 0 \\ 1/2 & 1 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix}$$

$$B^{-1}A^{-1} = \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1/4 & 0 \\ 1/2 & 1 \end{bmatrix} = \begin{bmatrix} -5/4 & -1 \\ 1 & 1 \end{bmatrix}$$

Theorem: If A is an invertible $n \times n$ matrix and B is an $n \times p$ matrix then the matrix equation $AX = B$ has the unique solution

$$X = A^{-1}B.$$

Proof: Observe that $A(A^{-1}B) = (AA^{-1})B = IB = B$.

Thus if $A\underline{x} = \underline{b}$ then $\underline{x} = A^{-1}\underline{b}$.

eg Solve $\begin{cases} 3x - y = 11 \\ x + y = -3. \end{cases}$

This is the same as the matrix equation

$$\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 11 \\ -3 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 11 \\ -3 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 8 \\ -20 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

So $\boxed{x = 2, \ y = -5}$.

Theorem: If A is an invertible matrix,

① A^n is invertible for any natural number n, and $(A^n)^{-1} = (A^{-1})^n$

② kA is invertible for any non-zero scalar k, and $(kA)^{-1} = \frac{1}{k}A^{-1}$

Proof: We will derive ②. We have

$$\begin{aligned}(kA)(\tfrac{1}{k}A^{-1}) &= (k\cdot\tfrac{1}{k})(AA^{-1}) \\ &= 1I \\ &= I.\end{aligned}$$

Thus $(kA)^{-1} = \frac{1}{k}A^{-1}$.

Theorem: Let A and B be matrices and k is a scalar. Then

① $(A+B)^T = A^T + B^T$

② $(kA)^T = kA^T$

③ $(A^T)^T = A$

④ if A is invertible, so is A^T and $(A^T)^{-1} = (A^{-1})^T$

⑤ $(AB)^T = B^T A^T$

Proof: We will obtain ⑤. Let A be an $m\times n$ matrix and B be an $n\times p$ matrix so

$$A = \begin{bmatrix} a_1^T \rightarrow \\ a_2^T \rightarrow \\ \vdots \\ a_m^T \rightarrow \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_1 & b_2 & \cdots & b_p \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}.$$

Then
$$AB = \begin{bmatrix} \underline{a}_1\cdot\underline{b}_1 & \underline{a}_1\cdot\underline{b}_2 & \cdots & \underline{a}_1\cdot\underline{b}_p \\ \underline{a}_2\cdot\underline{b}_1 & \underline{a}_2\cdot\underline{b}_2 & \cdots & \underline{a}_2\cdot\underline{b}_p \\ \vdots & & & \\ \underline{a}_m\cdot\underline{b}_1 & \underline{a}_m\cdot\underline{b}_2 & \cdots & \underline{a}_m\cdot\underline{b}_p \end{bmatrix}$$

so
$$(AB)^T = \begin{bmatrix} \underline{a}_1\cdot\underline{b}_1 & \underline{a}_2\cdot\underline{b}_1 & \cdots & \underline{a}_m\cdot\underline{b}_1 \\ \underline{a}_1\cdot\underline{b}_2 & \underline{a}_2\cdot\underline{b}_2 & & \underline{a}_m\cdot\underline{b}_2 \\ \vdots & \vdots & & \vdots \\ \underline{a}_1\cdot\underline{b}_p & \underline{a}_2\cdot\underline{b}_p & & \underline{a}_m\cdot\underline{b}_p \end{bmatrix}$$

Likewise, $A^T = \begin{bmatrix} \underline{a}_1 & \underline{a}_2 & \cdots & \underline{a}_m \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$ and $B^T = \begin{bmatrix} \underline{b}_1 \rightarrow \\ \underline{b}_2 \rightarrow \\ \vdots \\ \underline{b}_p \rightarrow \end{bmatrix}$

so
$$B^T A^T = \begin{bmatrix} \underline{b}_1\cdot\underline{a}_1 & \underline{b}_1\cdot\underline{a}_2 & \cdots & \underline{b}_1\cdot\underline{a}_m \\ \underline{b}_2\cdot\underline{a}_1 & \underline{b}_2\cdot\underline{a}_2 & \cdots & \underline{b}_2\cdot\underline{a}_m \\ \vdots & & & \\ \underline{b}_p\cdot\underline{a}_1 & \underline{b}_p\cdot\underline{a}_2 & \cdots & \underline{b}_p\cdot\underline{a}_m \end{bmatrix} = (AB)^T.$$

Def'n: An upper triangular matrix is a matrix in which all the entries below the diagonal are zero.

A lower triangular matrix is a matrix in which all the entries above the diagonal are zero.

eg $A = \begin{bmatrix} 3 & -2 & 5 \\ 0 & 7 & 6 \\ 0 & 0 & -1 \end{bmatrix}$ is upper triangular

Theorem : ① The product of upper triangular matrices is an upper triangular matrix, and the product of lower triangular matrices is a lower triangular matrix.

② If an upper triangular matrix is invertible then its inverse is an upper triangular matrix, and the inverse of a lower triangular matrix is a lower triangular matrix.

③ The transpose of an upper triangular matrix is a lower triangular matrix, and vice versa.

A diagonal matrix is both an upper triangular matrix and a lower triangular matrix.