Section 1.5: Euclidean n-Space

An <u>n-dimensional vector</u> has the form $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ where

$x_1, x_2, \ldots, x_n$ are the n components of the vector.

Most properties of 2- and 3-dimensional vectors still apply for $n \geq 4$.

Two vectors are <u>equal</u> if they have the same dimension and all corresponding components are equal.

Scalar multiplication is the same: $k \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} kx_1 \\ kx_2 \\ \vdots \\ kx_n \end{bmatrix}$.

Likewise, vector addition and vector subtraction work in the same way as long as all vectors being added or subtracted have the same dimension.

A linear combination of the m n-dimensional vectors $\underline{u}_1, \underline{u}_2, \dots, \underline{u}_m$ is a vector of the form

$$k_1 \underline{u}_1 + k_2 \underline{u}_2 + \dots + k_m \underline{u}_m$$

where $k_1, k_2, \dots, k_m$ are scalars.

To span Euclidean n-space, we need n n-dimensional vectors, none of which are parallel. However, this condition is necessary but not sufficient.

The standard basis vectors for n-space are $\underline{e}_1, \underline{e}_2, \dots, \underline{e}_n$ where $\underline{e}_i$ is an n-dimensional vectors with 1 as its ith component, and 0 for all other components.

eg In 5-space, $\quad \underline{e}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

Then we can write any vector in n-space as

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \underline{e}_1 + x_2 \underline{e}_2 + x_3 \underline{e}_3 + \dots + x_n \underline{e}_n$$

We denote Euclidean n-space as $\mathbb{R}^n$.

The dot product of two n-dimensional vectors $\underline{u} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

and $\underline{v} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ is $\underline{u} \cdot \underline{v} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$.

Recall that $\|\underline{u}\| = \sqrt{\underline{u} \cdot \underline{u}}$ so the norm of an n-dimensional vector is $\|\underline{u}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.

Because $\underline{u}\cdot\underline{v} = \|\underline{u}\|\,\|\underline{v}\|\cos(\theta)$, we can say that the angle θ between two n-dimensional vectors $\underline{u}$ and $\underline{v}$ is a number θ, $0 \le \theta \le \pi$ for which

$$\cos(\theta) = \frac{\underline{u}\cdot\underline{v}}{\|\underline{u}\|\,\|\underline{v}\|}.$$

The Cauchy-Schwarz inequality also still applies:

$$|\underline{u}\cdot\underline{v}| \le \|\underline{u}\|\,\|\underline{v}\|$$

Two n-dimensional vectors $\underline{u}$ and $\underline{v}$ are orthogonal if $\underline{u}\cdot\underline{v} = 0$.

eg Consider $\underline{u} = \begin{bmatrix} 5 \\ -2 \\ 0 \\ 7 \\ 1 \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} -1 \\ 4 \\ 2 \\ 2 \\ -1 \end{bmatrix}$. Then

$$\underline{u}\cdot\underline{v} = -5 - 8 + 0 + 14 - 1 = 0$$

so $\underline{u}$ and $\underline{v}$ are orthogonal.

<u>Theorem</u> : <u>The Triangle Inequality</u>

Given n-dimensional vectors $\underline{u}$ and $\underline{v}$ then $\|\underline{u}+\underline{v}\| \le \|\underline{u}\| + \|\underline{v}\|$

<u>Proof</u>: We can write $\|\underline{u}+\underline{v}\|^2 = (\underline{u}+\underline{v})\cdot(\underline{u}+\underline{v})$

$$= \underline{u}\cdot\underline{u} + \underline{u}\cdot\underline{v} + \underline{v}\cdot\underline{u} + \underline{v}\cdot\underline{v}$$

$$= \|\underline{u}\|^2 + 2\,\underline{u}\cdot\underline{v} + \|\underline{v}\|^2$$

If $\underline{u}$ and $\underline{v}$ are orthogonal then $\underline{u}\cdot\underline{v} = 0$ so

$$\|\underline{u}+\underline{v}\|^2 = \|\underline{u}\|^2 + \|\underline{v}\|^2$$

This is the Pythagorean theorem for vectors.

Otherwise, $\underline{u} \cdot \underline{v}$ is a non-zero number so we can write $\underline{u} \cdot \underline{v} \leq |\underline{u} \cdot \underline{v}|$ so

$$\|\underline{u}+\underline{v}\|^2 \leq \|\underline{u}\|^2 + 2|\underline{u} \cdot \underline{v}| + \|\underline{v}\|^2$$

$$\leq \|\underline{u}\|^2 + 2\|\underline{u}\|\|\underline{v}\| + \|\underline{v}\|^2$$

by the Cauchy-Schwarz inequality.

Now we can factor and write

$$\|\underline{u}+\underline{v}\|^2 \leq (\|\underline{u}\| + \|\underline{v}\|)^2$$

$$\|\underline{u}+\underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\|.$$

We already know that, given m n-dimensional vectors $\underline{u}_1, \underline{u}_2, \underline{u}_3, \ldots, \underline{u}_m$ then we can write

$$\underline{0} = 0\underline{u}_1 + 0\underline{u}_2 + 0\underline{u}_3 + \cdots + 0\underline{u}_m.$$

This is the <u>trivial</u> linear combination because all the scalar coefficients are 0. We are interested in finding whether there are <u>non-trivial</u> combinations of these vectors which still produce $\underline{0}$, that is, where not all the scalar coefficients are 0.

<u>Def'n</u>: A set of vectors $\underline{u}_1, \underline{u}_2, \ldots \underline{u}_m$ is <u>linearly independent</u> if $k_1\underline{u}_1 + k_2\underline{u}_2 + \cdots + k_m\underline{u}_m = \underline{0}$ implies $k_1 = k_2 = \cdots = k_m = 0$. Otherwise it is linearly <u>dependent</u>.

eg Determine whether $\underline{u} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ are linearly independent.

We set

$$k_1 \underline{u} + k_2 \underline{v} = \underline{0}$$

$$k_1 \begin{bmatrix} -4 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus $-4k_1 + 2k_2 = 0$

$k_1 - k_2 = 0 \longrightarrow k_1 = k_2$ so $-4k_1 + 2k_1 = 0$

$-2k_1 = 0$

$k_1 = 0$

$k_2 = 0$

We have only the trivial combination so $\underline{u}$ and $\underline{v}$ are linearly <u>independent</u>.

eg Determine whether $\underline{u} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$ are linearly independent.

We set

$$k_1 \underline{u} + k_2 \underline{v} = \underline{0}$$

$$k_1 \begin{bmatrix} -4 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 8 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus $-4k_1 + 8k_2 = 0$

$k_1 - 2k_2 = 0 \longrightarrow k_1 = 2k_2$ so $-4(2k_2) + 8k_2 = 0$

$-8k_2 + 8k_2 = 0$

$0 = 0$

This means that k_2 can be any value, say $k_2 = 1$ so $k_1 = 2$.

Hence

$$2\begin{bmatrix} -4 \\ 1 \end{bmatrix} + \begin{bmatrix} 8 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and so $\underline{u}$ and $\underline{v}$ are <u>linearly dependent</u>.

<u>Theorem</u>: Two vectors are linearly dependent if and only if they are parallel.

eg Determine whether $\underline{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\underline{v} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, $\underline{w} = \begin{bmatrix} -3 \\ -2 \\ -5 \end{bmatrix}$

are linearly independent.

We set

$$k_1 \underline{u} + k_2 \underline{v} + k_3 \underline{w} = \underline{0}$$

$$k_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + k_2 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + k_3 \begin{bmatrix} -3 \\ -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus $k_1 + 2k_2 - 3k_3 = 0$ $\qquad \rightarrow k_1 + 2(2k_3) - 3k_3 = 0$

$k_2 - 2k_3 = 0 \rightarrow k_2 = 2k_3$ $\qquad k_1 + k_3 = 0$

$-k_1 + 2k_2 - 5k_3 = 0$ $\qquad k_1 = -k_3$

Now $-(-k_3) + 2(2k_3) - 5k_3 = 0$

$k_3 + 4k_3 - 5k_3 = 0$

$0 = 0$ $\qquad$ We can set $k_3 = 1$

$k_2 = 2$

$k_1 = -1$

Hence $-\underline{u} + 2\underline{v} + \underline{w} = \underline{0}$ and so these are vectors are <u>linearly dependent</u>.

In the previous example, we have $-\underline{u} + 2\underline{v} + \underline{w} = \underline{0}$

so $\underline{w} = \underline{u} - 2\underline{v}$.

<u>Theorem</u>: Three or more vectors are linearly dependent if one of the vectors is a linear combination of the other vectors.

eg Determine whether $\underline{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\underline{v} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $\underline{w} = \begin{bmatrix} -3 \\ -2 \\ -5 \end{bmatrix}$

are linearly independent.

We set $k_1\underline{u} + k_2\underline{v} + k_3\underline{w} = \underline{0}$

$$k_1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + k_2\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + k_3\begin{bmatrix} -3 \\ -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus
$$\begin{aligned} k_1 \quad - 3k_3 &= 0 \rightarrow k_1 = 3k_3 \\ k_2 - 2k_3 &= 0 \rightarrow k_2 = 2k_3 \\ -k_1 + 2k_2 - 5k_3 &= 0 \rightarrow -(3k_3) + 2(2k_3) - 5k_3 = 0 \end{aligned}$$

$$-3k_3 + 4k_3 - 5k_3 = 0$$

$$-4k_3 = 0 \rightarrow k_3 = 0$$

$$k_2 = 0$$

$$k_1 = 0$$

Only the trivial solution is possible, so $\underline{u}, \underline{v}, \underline{w}$ are linearly <u>independent</u>.

<u>Theorem</u>: A set of n n-dimensional vectors span $\mathbb{R}^n$ if and only if they are linearly independent.