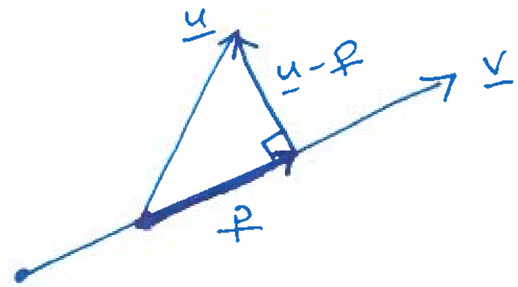
Section 1.4: Projections

Consider a vector $\underline{u}$ with its starting point ~~(scribbled out)~~ lying on a vector $\underline{v}$.



We draw a third vector that starts on $\underline{v}$, is orthogonal to $\underline{v}$, and shares its terminal point with $\underline{u}$.

This results in a vector starting at the start point of $\underline{u}$ and ending at the start point of the orthogonal vector, which we call the <u>projection</u> of $\underline{u}$ onto $\underline{v}$. We denote it by

$$\underline{p} = \text{proj}_{\underline{v}}\, \underline{u}$$

Thus the orthogonal vector is $\underline{u} - \underline{p}$.

Note that $\underline{p}$ and $\underline{v}$ are parallel, so $\underline{p} = k\underline{v}$.

But then
$$(\underline{u} - \underline{p}) \cdot \underline{v} = 0$$
$$(\underline{u} - k\underline{v}) \cdot \underline{v} = 0$$
$$\underline{u} \cdot \underline{v} - k\underline{v} \cdot \underline{v} = 0$$

$$k\underline{v} \cdot \underline{v} = \underline{u} \cdot \underline{v}$$

$$k = \frac{\underline{u} \cdot \underline{v}}{\underline{v} \cdot \underline{v}} \qquad \text{so } \underline{p} = \frac{\underline{u} \cdot \underline{v}}{\underline{v} \cdot \underline{v}}\, \underline{v}$$

eg Find the projection of $\underline{u} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ onto $\underline{v} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$.

We have $\underline{u} \cdot \underline{v} = 2 + 3 + 3 = 8$

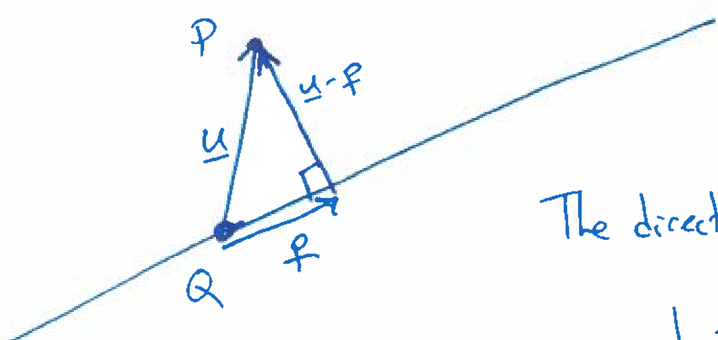
$$\underline{v} \cdot \underline{v} = 1 + 1 + 9 = 11$$

Thus $\underline{p} = \text{proj}_{\underline{v}}\, \underline{u} = \frac{8}{11} \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$

eg Find the distance from the point $P(1,3,-2)$ to the line with equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

We choose any point Q on the line, such as $(2,0,-1)$.



Let $\underline{u} = \overrightarrow{QP} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$.

The direction vector for the line is

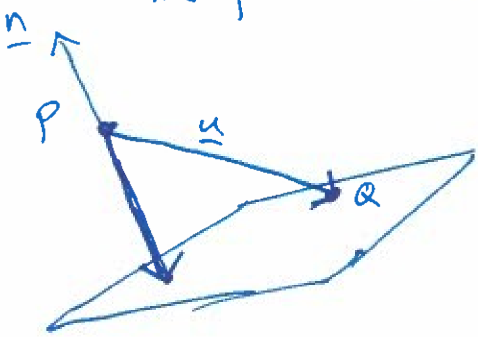
$$\underline{d} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Then $\underline{p} = \text{proj}_{\underline{d}}\, \underline{u} = \frac{\underline{u} \cdot \underline{d}}{\underline{d} \cdot \underline{d}}\, \underline{d} = \frac{-4}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$

So $\underline{u} - \underline{p} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $\|\underline{u} - \underline{p}\| = \sqrt{1+1+1} = \sqrt{3}$

Hence the distance from P to the line is $\boxed{\sqrt{3}}$

eg Find the distance from the point $P(2,-7,-7)$ to the plane with equation $3x-3y-2z=8$.



We choose any point Q in the plane, such as $(0,0,-4)$.

Then we have the vector $\underline{u} = \overrightarrow{PQ} = \begin{bmatrix} -2 \\ 7 \\ 3 \end{bmatrix}$.

The normal to the plane is $\underline{n} = \begin{bmatrix} 3 \\ -3 \\ -2 \end{bmatrix}$.

Let $\underline{p}$ be the projection of $\underline{u}$ onto $\underline{n}$, so its norm will be the distance from P to the plane. Thus

$$\underline{p} = \text{proj}_{\underline{n}}\, \underline{u} = \frac{\underline{u}\cdot\underline{n}}{\underline{n}\cdot\underline{n}}\,\underline{n} = \frac{-33}{22}\begin{bmatrix} 3 \\ -3 \\ -2 \end{bmatrix} = \frac{3}{2}\begin{bmatrix} -3 \\ 3 \\ 2 \end{bmatrix}.$$

$$\|\underline{p}\| = \frac{3}{2}\sqrt{9+9+4} \boxed{= \frac{3\sqrt{22}}{2}}$$

eg Find two orthogonal vectors in the plane $5x+y-z=0$.

First we find any two non-parallel vectors in the plane.

From the equation, we have $z = 5x+y$ so any vector in the plane has the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 5x+y \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 5x \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Hence $\underline{u} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ span the plane and are not parallel.

We can project one onto the other to get, say,

$$\underline{p} = \text{proj}_{\underline{v}}\, \underline{u} = \frac{\underline{u} \cdot \underline{v}}{\underline{v} \cdot \underline{v}}\, \underline{v} = \frac{5}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5/2 \\ 5/2 \end{bmatrix}$$

so that $\underline{u} - \underline{p} = \begin{bmatrix} 1 \\ -5/2 \\ 5/2 \end{bmatrix}$ is orthogonal to $\underline{v}$.

Thus two orthogonal vectors in the plane are $\begin{bmatrix} 1 \\ -5/2 \\ 5/2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Def'n: The projection of a vector $\underline{u}$ onto a plane Π is a vector $\underline{p}$ lying in Π such that $\underline{u} - \underline{p}$ is orthogonal to every vector in Π. We write

$$\underline{p} = \text{proj}_{\Pi}\, \underline{u}.$$

Theorem: Suppose that a plane Π is spanned by vectors $\underline{e}$ and $\underline{f}$. If a vector $\underline{u}$ is orthogonal to both $\underline{e}$ and $\underline{f}$ then it is orthogonal to every vector in Π.

Proof: Let $\underline{v}$ be any vector in Π so, for scalars k and ℓ, we can write

$$\underline{v} = k\underline{e} + \ell \underline{f}.$$

Then $\underline{u} \cdot \underline{e} = 0$ and $\underline{u} \cdot \underline{f} = 0$ so

$$\begin{aligned} \underline{u} \cdot \underline{v} &= \underline{u} \cdot (k\underline{e} + \ell \underline{f}) \\ &= k(\underline{u} \cdot \underline{e}) + \ell(\underline{u} \cdot \underline{f}) \\ &= k(0) + \ell(0) \\ &= 0 \end{aligned}$$

so $\underline{u}$ is orthogonal to $\underline{v}$.

So now consider a plane Π spanned by $\underline{e}$ and $\underline{f}$, and let $\underline{p} = \text{proj}_{\Pi}\, \underline{u}$. Since $\underline{p}$ is a vector in Π, there are scalars k, ℓ for which $\underline{p} = k\underline{e} + \ell \underline{f}$.

We must have $\underline{u} - \underline{p}$ orthogonal to both $\underline{e}$ and $\underline{f}$.

Then $(\underline{u} - \underline{p}) \cdot \underline{e} = 0$ and $(\underline{u} - \underline{p}) \cdot \underline{f} = 0$.

The first of these can be written $\underline{u} \cdot \underline{e} - \underline{p} \cdot \underline{e} = 0$

$$\underline{u} \cdot \underline{e} = \underline{p} \cdot \underline{e}$$

$$\underline{u} \cdot \underline{e} = (k\underline{e} + \ell \underline{f}) \cdot \underline{e}$$

$$= k\underline{e} \cdot \underline{e} + \ell\, \underline{f} \cdot \underline{e}$$

Now suppose $\underline{e}$ and $\underline{f}$ are orthogonal spanning vectors, so $\underline{f} \cdot \underline{e} = 0$. So now $\underline{u} \cdot \underline{e} = k\underline{e} \cdot \underline{e}$ and $k = \dfrac{\underline{u} \cdot \underline{e}}{\underline{e} \cdot \underline{e}}$.

Likewise $\ell = \dfrac{\underline{u} \cdot \underline{f}}{\underline{f} \cdot \underline{f}}$. Hence

$$\text{proj}_{\Pi}\, \underline{u} = \frac{\underline{u} \cdot \underline{e}}{\underline{e} \cdot \underline{e}}\, \underline{e} + \frac{\underline{u} \cdot \underline{f}}{\underline{f} \cdot \underline{f}}\, \underline{f}$$

where $\underline{e}$ and $\underline{f}$ are orthogonal vectors that span Π.

eg Find the projection of $\underline{u} = \begin{bmatrix} -4 \\ -2 \\ 1 \end{bmatrix}$ onto the plane with equation $5x + y - z = 0$.

We have already found two orthogonal vectors in this plane to be $\underline{e} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\underline{f} = \begin{bmatrix} 2 \\ -5 \\ 5 \end{bmatrix}$.

So then $\underline{u} \cdot \underline{e} = -1$ $\qquad \underline{e} \cdot \underline{e} = 2$

$\underline{u} \cdot \underline{f} = 7$ $\qquad \underline{f} \cdot \underline{f} = 54$

Hence $\operatorname{proj}_{\pi} \underline{u} = \frac{-1}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \frac{7}{54} \begin{bmatrix} 2 \\ -5 \\ 5 \end{bmatrix}$

$$\boxed{= \frac{1}{27} \begin{bmatrix} 7 \\ -31 \\ 4 \end{bmatrix}}.$$