1. a) $\|\underline{u}\| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$

A unit vector is $\frac{\underline{u}}{\|\underline{u}\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \frac{\sqrt{6}}{6} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

A unit vector in the opposite direction to $\underline{u}$ is

$$-\frac{\sqrt{6}}{6} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

b) $\underline{u} \cdot \underline{v} = \|\underline{u}\| \|\underline{v}\| \cos(\theta)$

$\underline{u} \cdot \underline{v} = 2(1) + (-1)(1) + 1(2) = 3$

$\|\underline{v}\| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$

Then $3 = \sqrt{6} \cdot \sqrt{6} \cos(\theta)$

$\cos(\theta) = \frac{3}{6} = \frac{1}{2} \rightarrow \boxed{\theta = \frac{\pi}{3}}$

c) We set $(\underline{u} + k\underline{v}) \cdot (\underline{u} - k\underline{v}) = 0$

IN THE VIDEO, I MISSPOKE AND DESCRIBED THIS AS "$k\underline{u}$" BUT IT SHOULD BE "$k\underline{v}$"

$\underline{u} \cdot \underline{u} + k\underline{v} \cdot \underline{u} - \underline{u} \cdot k\underline{v} - (k\underline{v}) \cdot (k\underline{v}) = 0$

$\underline{u} \cdot \underline{u} + k(\underline{u} \cdot \underline{v}) - k(\underline{u} \cdot \underline{v}) - k^2(\underline{v} \cdot \underline{v}) = 0$

$\|\underline{u}\|^2 - k^2 \|\underline{v}\|^2 = 0$

$(\sqrt{6})^2 - k^2 (\sqrt{6})^2 = 0$

$k^2 = \frac{6}{6} = 1$

$\boxed{k = \pm 1}$

2. a) $\overrightarrow{AB} = \begin{bmatrix} -2-1 \\ 2-(-1) \\ 2-2 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 0 \end{bmatrix}$

$\overrightarrow{AC} = \begin{bmatrix} -1-1 \\ 0-(-1) \\ 1-2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$

$$\underline{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} 3 & 1 \\ 0 & -1 \end{vmatrix} \hat{\imath} - \begin{vmatrix} -3 & -2 \\ 0 & -1 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} -3 & -2 \\ 3 & 1 \end{vmatrix} \hat{k}$$

$$= -3\hat{\imath} - 3\hat{\jmath} + 3\hat{k}$$

$$= \begin{bmatrix} -3 \\ -3 \\ 3 \end{bmatrix}$$

The equation of Π has the form

$$-3x - 3y + 3z = d$$

Using $C(-1,0,1)$ we have

$$-3(-1) - 3(0) + 3(1) = d$$

$$6 = d$$

and so Π has the equation

$$-3x - 3y + 3z = 6$$

$$\boxed{-x - y + z = 2}$$

2.b) The direction vector for ℓ is

$$\underline{d} = \underline{n} = \begin{bmatrix} -3 \\ -3 \\ 3 \end{bmatrix}$$

So ℓ has equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + t\underline{d} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + t \begin{bmatrix} -3 \\ -3 \\ 3 \end{bmatrix}$$

Thus $x = 2 - 3t$

$y = 3 - 3t$

$z = 4 + 3t$

The equation of the plane is

$$-x - y + z = 2$$

$$-(2-3t) - (3-3t) + (4+3t) = 2$$

$$-2 + 3t - 3 + 3t + 4 + 3t = 2$$

$$9t - 1 = 2$$

$$9t = 3 \rightarrow t = 1/3$$

Thus $x = 2 - 3 \cdot 1/3 = 2 - 1 = 1$

$y = 3 - 3 \cdot 1/3 = 3 - 1 = 2$

$z = 4 + 3 \cdot 1/3 = 4 + 1 = 5$

The point of intersection is $(1, 2, 5)$.

2. c) $\overrightarrow{PC} = \begin{bmatrix} -1-3 \\ 0-2 \\ 1-1 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 0 \end{bmatrix}$ $\qquad \underline{n} = \begin{bmatrix} -3 \\ -3 \\ 3 \end{bmatrix}$

$$\underline{p} = \text{proj}_{\underline{n}} \overrightarrow{PC} = \frac{\overrightarrow{PC} \cdot \underline{n}}{\underline{n} \cdot \underline{n}} \underline{n}$$

$$= \frac{(-4)(-3) + (-2)(-3) + 0(3)}{(-3)^2 + (-3)^2 + 3^2} \begin{bmatrix} -3 \\ -3 \\ 3 \end{bmatrix}$$

$$= \frac{18}{27} \begin{bmatrix} -3 \\ -3 \\ 3 \end{bmatrix}$$

$$= \frac{2}{3} \begin{bmatrix} -3 \\ -3 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix}$$

$$\|\underline{p}\| = \sqrt{(-2)^2 + (-2)^2 + 2^2} = \sqrt{12} = \boxed{2\sqrt{3}}$$

Observe that $\underline{p} = \overrightarrow{PQ}$ so, if Q is the point (x,y,z) then

$$\begin{bmatrix} x-3 \\ y-2 \\ z-1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix}$$

so $x-3 = -2 \rightarrow x = 1$

$y-2 = -2 \rightarrow y = 0$

$z-1 = 2 \rightarrow z = 3$ $\qquad$ so Q is the point $(1,0,3)$.

3.

$$\left(A^T - 3\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A^T - 3\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$= \frac{1}{2-1}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 6 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -1 \\ 5 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & -1 \\ 5 & -1 \end{bmatrix}^T = \begin{bmatrix} 4 & 5 \\ -1 & -1 \end{bmatrix}$$

4. $\left[\begin{array}{ccc|c} 0 & 1 & 2k & 0 \\ 1 & 2 & 6 & 2 \\ k & 0 & 2 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 2 & 6 & 2 \\ 0 & 1 & 2k & 0 \\ k & 0 & 2 & 1 \end{array}\right]$

$\xrightarrow{R_3 \to R_3 - kR_1} \left[\begin{array}{ccc|c} 1 & 2 & 6 & 2 \\ 0 & 1 & 2k & 0 \\ 0 & -2k & 2-6k & 1-2k \end{array}\right] \xrightarrow{R_3 \to R_3 + 2kR_2} \left[\begin{array}{ccc|c} 1 & 2 & 6 & 2 \\ 0 & 1 & 2k & 0 \\ 0 & 0 & 2-6k+4k^2 & 1-2k \end{array}\right]$

a) $2-6k+4k^2=0$

$2k^2-3k+1=0$

$(2k-1)(k-1)=0$

$k=\frac{1}{2}, \ k=1$

$1-2k=0$

$k=\frac{1}{2}$

There will be an infinite number of solutions if $k=\frac{1}{2}$.

b) There will be no solutions if $k=1$.

c) There will be a unique solution if $k \neq \frac{1}{2}$ and $k \neq 1$.

5. a) $\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 2 & 1 & 2 \\ -2 & 2 & 1 & -4 & 0 & -7 \\ 1 & -1 & 1 & 3 & 1 & -1 \end{array}\right]$

$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 + 2R_1} \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 1 & 1 & 0 & -3 \end{array}\right]$

$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -2 & 0 \end{array}\right]$

Let $x_2 = t$, $x_5 = s$

$x_4 = 2x_5 = 2s$

$x_3 = -3 - 2x_5 = -3 - 2s$

$x_1 = 2 - x_5 - 2x_4 + x_2 = 2 - s - 2(2s) + t = 2 - 5s + t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2-5s+t \\ t \\ -3-2s \\ 2s \\ s \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -5 \\ 0 \\ -2 \\ 2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

b) $\underline{x}_h = s \begin{bmatrix} -5 \\ 0 \\ -2 \\ 2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

because the solution in part (a) has the form $\underline{x} = \underline{x}_p + \underline{x}_h$

c) From part (a),

$$\begin{bmatrix} 2 \\ -7 \\ -1 \end{bmatrix} = 2\underline{v}_1 + 0\underline{v}_2 - 3\underline{v}_3 + 0\underline{v}_4 + 0\underline{v}_5$$

d) Because there is an infinite number of solutions to the corresponding homogeneous system of equations, these vectors are linearly <u>dependent</u>.

6. a)

$$\left[\begin{array}{ccc|ccc} -1 & 2 & -3 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to (-1)R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & -1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & -2 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & -1 & 0 & 0 \\ 0 & 5 & -6 & 2 & 1 & 0 \\ 0 & 6 & -7 & 4 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{5} R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & -1 & 0 & 0 \\ 0 & 1 & -6/5 & 2/5 & 1/5 & 0 \\ 0 & 6 & -7 & 4 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - 6R_2]{R_1 \to R_1 + 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3/5 & -1/5 & 2/5 & 0 \\ 0 & 1 & -6/5 & 2/5 & 1/5 & 0 \\ 0 & 0 & 1/5 & 8/5 & -6/5 & 1 \end{array}\right] \xrightarrow{R_3 \to 5R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 3/5 & -1/5 & 2/5 & 0 \\ 0 & 1 & -6/5 & 2/5 & 1/5 & 0 \\ 0 & 0 & 1 & 8 & -6 & 5 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 + \frac{6}{5} R_3]{R_1 \to R_1 - \frac{3}{5} R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 4 & -3 \\ 0 & 1 & 0 & 10 & -7 & 6 \\ 0 & 0 & 1 & 8 & -6 & 5 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{bmatrix}$$

b)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -6 \\ 14 \\ 11 \end{bmatrix}$$

7. $\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 3 \\ 0 & -5 \end{bmatrix} \xrightarrow{R_2 \to -\frac{1}{5}R_2} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

$\xrightarrow{R_1 \to R_1 - 3R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$E_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1/5 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$

$$E_3 E_2 E_1 A = I$$

$$A = E_1^{-1} E_2^{-1} E_3^{-1}$$

$E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \quad E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -5 \end{bmatrix} \quad E_3^{-1} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

8. a) $\det(A) = -2\begin{vmatrix} 2 & -2 & 1 \\ -1 & 2 & 0 \\ -4 & 3 & -2 \end{vmatrix} + 0 - 0 + 4\begin{vmatrix} 1 & 2 & -2 \\ 3 & -1 & 2 \\ 0 & -4 & 3 \end{vmatrix}$

$$\begin{vmatrix} 2 & -2 & 1 \\ -1 & 2 & 0 \\ -4 & 3 & -2 \end{vmatrix} = 1\begin{vmatrix} -2 & 1 \\ 3 & -2 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 \\ -4 & -2 \end{vmatrix} - 0$$

$$= 1 + 2(0) = 1$$

$$\begin{vmatrix} 1 & 2 & -2 \\ 3 & -1 & 2 \\ 0 & -4 & 3 \end{vmatrix} = 0 - (-4)\begin{vmatrix} 1 & -2 \\ 3 & 2 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}$$

$$= 4(8) + 3(-7) = 11$$

$\det(A) = -2 \cdot 1 + 4 \cdot 11 \boxed{= 42}$

b) $\det(A^2 B^T A^{-1}) = \det(A^2)\det(B^T)\det(A^{-1})$

$$= [\det(A)]^2 \det(B) \cdot \frac{1}{\det(A)}$$

$$= \det(A)\det(B)$$

$$= 2\det(B)$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{R_1 \to R_1 + R_2} \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix} \xrightarrow{R_2 \to 2R_2} \begin{bmatrix} a+c & b+d \\ 2c & 2d \end{bmatrix} = B^T$$

$\det(B) = \det(B^T) = 2\det(A) = 2 \cdot 2 = 4$

Thus $\det(A^2 B^T A^{-1}) = 2 \cdot 4 \boxed{= 8}$

9. a) $A - \lambda I = \begin{bmatrix} -2-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix}$

The characteristic polynomial is

$$\det(A - \lambda I) = (-2-\lambda)(1-\lambda) - 4$$
$$= -2 + 2\lambda - \lambda + \lambda^2 - 4$$
$$= \lambda^2 + \lambda - 6$$

We set $\lambda^2 + \lambda - 6 = 0$

$$(\lambda+3)(\lambda-2) = 0$$
$$\lambda = -3, \ \lambda = 2$$

b) $\underline{\lambda = -3}$

$$A - \lambda I = A + 3I = \begin{bmatrix} 1 & 1 \\ 4 & 4 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 4R_1} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{array}{l} x_2 = t \\ x_1 = -x_2 = -t \end{array}$$

$\underline{x} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ so an eigenvector is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\underline{\lambda = 2}$

$$A - \lambda I = A - 2I = \begin{bmatrix} -4 & 1 \\ 4 & -1 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix} -4 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{array}{l} x_2 = t \\ x_1 = \frac{1}{4} x_2 = \frac{1}{4} t \end{array}$$

$\underline{x} = t \begin{bmatrix} 1/4 \\ 1 \end{bmatrix} = t \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ so an eigenvector is $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

c) $P = \begin{bmatrix} -1 & 1 \\ 1 & 4 \end{bmatrix}$

$D = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$

10. $A-\lambda I = \begin{bmatrix} 2-\lambda & -1 \\ 3 & 2-\lambda \end{bmatrix}$

$$\det(A-\lambda I) = (2-\lambda)^2+3$$
$$= 4-4\lambda+\lambda^2+3$$
$$= \lambda^2-4\lambda+7$$

We set $\lambda^2-4\lambda+7=0$

$$\lambda = \frac{4\pm\sqrt{16-28}}{2} = \frac{4\pm\sqrt{-12}}{2}$$
$$= \frac{4\pm 2\sqrt{3}i}{2} = 2\pm\sqrt{3}i$$

$\lambda_1 = 2+\sqrt{3}i$

$\lambda_2 = 2-\sqrt{3}i$

$$\frac{\lambda_1}{\lambda_2} = \frac{2+\sqrt{3}i}{2-\sqrt{3}i}\cdot\frac{2+\sqrt{3}i}{2+\sqrt{3}i} = \frac{4+2\sqrt{3}i+2\sqrt{3}i+3i^2}{4-3i^2}$$
$$= \frac{1+4\sqrt{3}i}{7} = \frac{1}{7}+\frac{4\sqrt{3}}{7}i$$

$$\frac{\lambda_2}{\lambda_1} = \frac{2-\sqrt{3}i}{2+\sqrt{3}i}\cdot\frac{2-\sqrt{3}i}{2-\sqrt{3}i} = \frac{4-2\sqrt{3}i-2\sqrt{3}i+3i^2}{4-3i^2}$$
$$= \frac{1-4\sqrt{3}i}{7} = \frac{1}{7}-\frac{4\sqrt{3}}{7}i$$

11. a) Let $\underline{u}$ be any vector in Π. Then $\underline{u} = k\underline{e} + l\underline{f}$ for scalars k and l.

We assume that $\underline{v}\cdot\underline{e} = 0$ and $\underline{v}\cdot\underline{f} = 0$.

We want to show that $\underline{v}\cdot\underline{u} = 0$.

We have
$$\begin{aligned}\underline{v}\cdot\underline{u} &= \underline{v}\cdot(k\underline{e} + l\underline{f}) \\ &= k\underline{v}\cdot\underline{e} + l\underline{v}\cdot\underline{f} \\ &= k(0) + l(0) \\ &= 0\end{aligned}$$
so $\underline{v}$ is orthogonal to every vector in Π.

b) We want to show that $AA = I$.

We have
$$\begin{aligned}AA &= (I-2E)(I-2E) \\ &= I^2 - 2EI - I(2E) + (2E)(2E) \\ &= I^2 - 2EI - 2IE + 4E^2 \\ &= I - 2E - 2E + 4E^2 \\ &= I - 4E + 4E^2 \\ &= I - 4E + 4E \quad \text{because } E^2 = E \\ &= I\end{aligned}$$
so $A^{-1} = A$

c) We are given that $A\underline{x} = \lambda\underline{x}$ where $\underline{x}$ is an eigenvector.

Then if $A^2 = I$, we have
$$\begin{aligned}A^2\underline{x} &= I\underline{x} \\ A(A\underline{x}) &= \underline{x} \\ A(\lambda\underline{x}) &= \underline{x} \\ \lambda A\underline{x} &= \underline{x} \\ \lambda(\lambda\underline{x}) &= \underline{x}\end{aligned}$$
$\rightarrow \lambda^2\underline{x} = \underline{x}$

Thus $\lambda^2 = 1$

$\lambda = \pm 1$.