MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTIONS 2.8 & 2.9 | **Math 2000 Worksheet** | WINTER 2020

SOLUTIONS

1. (a) The region of integration is defined by $0 \le x \le \sqrt{16-y^2}$ and $-4 \le y \le 4$, which is the semicircle centered on the origin with radius 4 lying in the positive x-plane. In polar coordinates this is equivalent to $0 \le r \le 4$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. As well, the function $\sqrt{x^2+y^2+9} = \sqrt{r^2+9}$. The integral can thus be written

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^4 \sqrt{r^2+9}r\,dr\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{1}{3}(r^2+9)^{\frac{3}{2}}\right]_0^4 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{98}{3}\,d\theta = \left[\frac{98}{3}\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{98\pi}{3}.$$

(b) First we have

$$x^2+(y-1)^2 = 1 \quad \Longrightarrow \quad x^2+y^2 = 2y \quad \Longrightarrow \quad r^2 = 2r\sin(\theta) \quad \Longrightarrow \quad r = 2\sin(\theta).$$

Furthermore, the entire circle is traced out for values of θ ranging from 0 to π. So in polar coordinates, D is defined by $0 \le \theta \le \pi$ and $0 \le r \le 2\sin(\theta)$. Also, $\sqrt{x^2+y^2} = r$. Thus, the integral becomes

$$\begin{aligned}
&\int_0^{\pi} \int_0^{2\sin(\theta)} r^2\,dr\,d\theta \\
&= \int_0^{\pi} \left[\frac{1}{3}r^3\right]_0^{2\sin(\theta)} d\theta = \int_0^{\pi} \frac{8}{3}\sin^3\theta\,d\theta = \frac{8}{3}\int_0^{\pi} [1-\cos^2(\theta)]\sin(\theta)\,d\theta \\
&= \frac{8}{3}\int_0^{\pi} [\sin(\theta) - \cos^2(\theta)\sin(\theta)]\,d\theta = \frac{8}{3}\left[-\cos(\theta) + \frac{1}{3}\cos^3(\theta)\right]_0^{\pi} = \frac{32}{9}.
\end{aligned}$$

(c) The line $y = x$ is the same as the polar function $\theta = \frac{\pi}{4}$. (If this is not intuitively obvious, note that $y = x$ means $r\sin(\theta) = r\cos(\theta)$ so $\sin(\theta) = \cos(\theta)$ and $\tan(\theta) = 1$, leading to the same conclusion.) Furthermore, D lies between the circles centered at the origin with radius 3 and 5. Hence D is defined by $0 \le \theta \le \frac{\pi}{4}$ and $3 \le r \le 5$. Also,

$$\frac{y^2}{x^2} = \frac{r^2\sin^2(\theta)}{r^2\cos^2(\theta)} = \tan^2(\theta).$$

The integral thus becomes

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \int_3^5 r\tan^2(\theta)\,dr\,d\theta &= \int_0^{\frac{\pi}{4}} \left[\frac{1}{2}r^2\tan^2(\theta)\right]_3^5 d\theta = \int_0^{\frac{\pi}{4}} 8\tan^2(\theta)\,d\theta \\
&= 8\int_0^{\frac{\pi}{4}} [\sec^2(\theta) - 1]\,d\theta = 8\Big[\tan(\theta) - \theta\Big]_0^{\frac{\pi}{4}} = 8\left[1 - \frac{\pi}{4}\right] = 8 - 2\pi.
\end{aligned}$$

2. (a) First we solve for the points of intersection of $y = x$ and $y = x^2$: $x = x^2 \implies x(x-1) = 0$ so $x = 0$ and $x = 1$. Thus the region of integration is defined by $0 \le x \le 1$ and $x^2 \le y \le x$. The integral is

$$V = \int_0^1 \int_{x^2}^{x} (1 - xy)\, dy dx = \int_0^1 \left[y - \frac{1}{2}xy^2 \right]_{x^2}^{x} dx = \int_0^1 \left[x - \frac{1}{2}x^3 - x^2 + \frac{1}{2}x^5 \right] dx$$
$$= \left[\frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{3}x^3 + \frac{1}{12}x^6 \right]_0^1 = \frac{1}{8}.$$

(b) First we find the points of intersection of $x = y$ and $x = y^2 - y$:

$$y = y^2 - y \quad \implies \quad y^2 - 2y = 0 \quad \implies \quad y(y-2) = 0$$

so $y = 0$ and $y = 2$. Then the region of integration is defined by $0 \le y \le 2$ and $y^2 - y \le x \le y$ so the integral is

$$V = \int_0^2 \int_{y^2-y}^{y} (3x^2 + y^2)\, dx dy = \int_0^2 \left[x^3 + xy^2 \right]_{y^2-y}^{y} dy = \int_0^2 [-y^6 + 3y^5 - 4y^4 + 4y^3]\, dy$$
$$= \left[-\frac{1}{7}y^7 + \frac{1}{2}y^6 - \frac{4}{5}y^5 + y^4 \right]_0^2 = \frac{144}{35}.$$

(c) To find the points of intersection of $y = x^2$ and $x = y^2$, we substitute the former into the latter to get

$$x = x^4 \quad \implies \quad x^4 - x = 0 \quad \implies \quad x(x^3 - 1) = 0$$

so $x = 0$ and $x = 1$. Also, we can write the function $x = y^2$ as $y = \sqrt{x}$ or $y = -\sqrt{x}$; however, the region of integration is bouned above only by the former. Hence the region is defined by $0 \le x \le 1$ and $x^2 \le y \le \sqrt{x}$. The volume of S is

$$V = \int_0^1 \int_{x^2}^{\sqrt{x}} xy\, dy dx = \int_0^1 \left[\frac{1}{2}xy^2 \right]_{x^2}^{\sqrt{x}} dx = \int_0^1 \left[\frac{1}{2}x^2 - \frac{1}{2}x^5 \right] dx$$
$$= \left[\frac{1}{6}x^3 - \frac{1}{12}x^6 \right]_0^1 = \frac{1}{12}.$$

(d) Due to the curved nature of the region of integration, we should use polar coordinates. Then the region is defined by $0 \le \theta \le 2\pi$ and $2 \le r \le 5$. Furthermore, the function $\sqrt{x^2 + y^2} = \sqrt{r^2} = r$. Thus

$$V = \int_0^{2\pi} \int_2^5 r^2\, dr\, d\theta = \int_0^{2\pi} \left[\frac{1}{3}r^3 \right]_2^5 d\theta = \int_0^{2\pi} 39\, d\theta = \left[39\theta \right]_0^{2\pi} = 78\pi.$$

(e) The equations of the lines bounding the triangle are $y = 1$, $x = 2y - 1$ and $x = 5 - y$ so the region of integration is defined by $1 \le y \le 2$ and $2y - 1 \le x \le 5 - y$. Then we have

$$V = \int_1^2 \int_{2y-1}^{5-y} (1 + xy)\, dx dy = \int_1^2 \left[x + \frac{1}{2}x^2 y \right]_{2y-1}^{5-y} dy = \int_1^2 \left[6 + 9y - 3y^2 - \frac{3}{2}y^3 \right] dy$$
$$= \left[6y + \frac{9}{2}y^2 - y^3 - \frac{3}{8}y^4 \right]_1^2 = \frac{55}{8}.$$

(f) The region of integration will be the intersection of the paraboloid with the xy-plane (that is, the plane defined by the equation $z = 0$), which is the curve defined by $4 - x^2 - y^2 = 0$, or $x^2 + y^2 = 4$. Consequently, it makes sense to use polar coordinates; as such, the region of integration is defined by $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 2$. The function $4 - x^2 - y^2 = 4 - r^2$, so then

$$V = \int_0^{2\pi} \int_0^2 (4 - r^2) r \, dr \, d\theta = \int_0^{2\pi} \int_0^2 (4r - r^3) \, dr \, d\theta = \int_0^{2\pi} \left[2r^2 - \frac{1}{4} r^4 \right]_0^2 d\theta$$

$$= \int_0^{2\pi} 4 \, d\theta = \left[4\theta \right]_0^{2\pi} = 8\pi.$$