MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SOLUTIONS

1. (a) We may integrate with respect to x and y in either order, for instance:

$$\iint_D \frac{1}{\sqrt{16-x^2}}\,dA = \int_{-2}^{2}\int_0^7 \frac{1}{\sqrt{16-x^2}}\,dy\,dx = \int_{-2}^{2}\left[\frac{y}{\sqrt{16-x^2}}\right]_0^7 dx$$
$$= \int_{-2}^{2} \frac{7}{\sqrt{16-x^2}}\,dx = \left[7\arcsin\left(\frac{x}{4}\right)\right]_{-2}^{2} = \frac{7\pi}{3}.$$

(b) We can write

$$D = \{(x,y)\,|\,0 \le x \le 3, 0 \le y \le x\}.$$

Then the integral becomes

$$\iint_D \frac{1}{\sqrt{16-x^2}}\,dA = \int_0^3\int_0^x \frac{1}{\sqrt{16-x^2}}\,dy\,dx = \int_0^3\left[\frac{y}{\sqrt{16-x^2}}\right]_0^x = \int_0^3 \frac{x}{\sqrt{16-x^2}}\,dx$$
$$= \left[-\sqrt{16-x^2}\right]_0^3 = 4-\sqrt{7}.$$

2. (a) $$\int_2^4\int_1^{\sqrt{y}} x(y^2-5y)\,dx\,dy = \int_2^4\left[x^2(y^2-5y)\right]_1^{\sqrt{y}} dy = \int_2^4 \frac{1}{2}(y^2-5y)(y-1)\,dy$$
$$= \int_2^4\left[\frac{1}{2}y^3-3y^2+\frac{5}{2}y\right]dy = \left[\frac{1}{8}y^4-y^3+\frac{5}{4}y^2\right]_2^4 = -11$$

(b) We have

$$\int_0^1\int_0^{y^2} \frac{y}{x^2+y^2}\,dx\,dy = \int_0^1\left[\frac{y}{y}\arctan\left(\frac{x}{y}\right)\right]_0^{y^2} dy = \int_0^1\left[\arctan\left(\frac{x}{y}\right)\right]_0^{y^2} dy$$
$$= \int_0^1 \arctan(y)\,dy = \left[y\arctan(y) - \frac{1}{2}\ln|1+y^2|\right]_0^1$$
$$= \frac{\pi}{4}-\frac{1}{2}\ln(2),$$

where the latter integral can be evaluated using integration by parts.

(c) We have

$$\int_1^{\sqrt[4]{10}}\int_0^x y^2\sqrt{x^4-1}\,dy\,dx = \int_1^{\sqrt[4]{10}}\left[\frac{1}{3}y^3\sqrt{x^4-1}\right]_0^x dx = \int_1^{\sqrt[4]{10}}\left[\frac{1}{3}x^3\sqrt{x^4-1}\right]dx$$
$$= \left[\frac{1}{18}(x^4-1)^{\frac{3}{2}}\right]_1^{\sqrt[4]{10}} = \frac{3}{2},$$

where the second integral can be evaluated using u-substitution.

(d) We have

$$\int_{\frac{\pi}{2}}^{0}\int_{0}^{\sin(x)} e^{\cos(x)}\,dy\,dx = \int_{\frac{\pi}{2}}^{0}\left[ye^{\cos(x)}\right]_0^{\sin(x)} dx = \int_{\frac{\pi}{2}}^{0} \sin(x)e^{\cos(x)}\,dx$$
$$= \left[-e^{\cos(x)}\right]_{\frac{\pi}{2}}^{0} = 1 - e,$$

where again we may use u-substitution to evaluate the final integral.

3. (a) The region of integration can be written as $0 \le y \le x$ and $0 \le x \le \sqrt{\pi}$. Hence the integral becomes

$$\int_0^{\sqrt{\pi}}\int_0^{x} \sin(x^2)\,dy\,dx = \int_0^{\sqrt{\pi}}\left[y\sin(x^2)\right]_0^x dx = \int_0^{\sqrt{\pi}} x\sin(x^2)\,dx$$
$$= \left[-\frac{1}{2}\cos(x^2)\right]_0^{\sqrt{\pi}} = 1.$$

(b) The region of integration can be written as $0 \le x \le \sqrt{y}$ and $0 \le y \le 9$. Hence the integral becomes

$$\int_0^9\int_0^{\sqrt{y}} xe^{y^2}\,dx\,dy = \int_0^9\left[\frac{1}{2}x^2e^{y^2}\right]_0^{\sqrt{y}} dy = \int_0^9 \frac{1}{2}ye^{y^2}\,dy = \left[\frac{1}{4}e^{y^2}\right]_0^9 = \frac{1}{4}e^{81} - \frac{1}{4}.$$

(c) The region of integration can be written as $0 \le y \le 2x$ and $0 \le x \le 2$. Hence the integral becomes

$$\int_0^2\int_0^{2x} \frac{y}{x^3+1}\,dy\,dx = \int_0^2\left[\frac{y^2}{2(x^3+1)}\right]_0^{2x} dx = \int_0^2 \frac{2x^2}{x^3+1}\,dx$$
$$= \left[\frac{2}{3}\ln|x^3+1|\right]_0^2 = \frac{4}{3}\ln(3).$$

(d) The region of integration can be written as $0 \le x \le \sin(y)$ and $0 \le y \le \frac{\pi}{2}$. Hence the integral becomes

$$\int_0^{\frac{\pi}{2}}\int_0^{\sin(y)} \sqrt{1+\cos(y)}\,dx\,dy = \int_0^{\frac{\pi}{2}}\left[x\sqrt{1+\cos(y)}\right]_0^{\sin(y)} dy$$
$$= \int_0^{\frac{\pi}{2}} \sin(y)\sqrt{1+\cos(y)}\,dy$$
$$= \left[-\frac{2}{3}[1+\cos(y)]^{\frac{3}{2}}\right]_0^{\frac{\pi}{2}} = \frac{4\sqrt{2}-2}{3}.$$

4. (a) First we need to solve for the points of intersection of the two curves. We set

$$x^2 + 2x = 24 - x^2 \quad\Longrightarrow\quad 2(x+4)(x-3) = 0$$

so $x = -4$ or $x = 3$. Then the region D is bounded by $-4 \leq x \leq 3$ and $x^2 + 2x \leq y \leq 24 - x^2$. (The order of the latter inequality may be easily checked by graphing, or by substitution of a value of x in the interval $(-4, 3)$, such as $x = 0$.) Then we have

$$\begin{aligned} A = \iint\limits_D dA &= \int_{-4}^{3} \int_{x^2+2x}^{24-x^2} dy\, dx \\ &= \int_{-4}^{3} \Big[y \Big]_{x^2+2x}^{24-x^2} dx \\ &= \int_{-4}^{3} [24 - 2x - 2x^2]\, dx \\ &= \left[24x - x^2 - \frac{2}{3}x^3 \right]_{-4}^{3} \\ &= \frac{343}{3}. \end{aligned}$$

(b) Again we begin by solving for the points of intersection. First we note that $y = 9 - 3x$ can also be written as $x = 3 - \frac{1}{3}y$. Then we have

$$\sqrt{9-y} = 3 - \frac{1}{3}y \quad \Longrightarrow \quad 9 - y = 9 - 2y + \frac{1}{9}y^2 \quad \Longrightarrow \quad y\left(\frac{1}{9}y - 1\right) = 0$$

so $y = 0$ or $y = 9$. Thus the region R is defined by $0 \leq y \leq 9$ and $3 - \frac{1}{3}y \leq x \leq \sqrt{9-y}$. Then we have

$$\begin{aligned} A = \iint\limits_R dA &= \int_{0}^{9} \int_{\frac{1}{3}y}^{\sqrt{9-y}} dx\, dy \\ &= \int_{0}^{9} \Big[x \Big]_{\frac{1}{3}y}^{\sqrt{9-y}} dy \\ &= \int_{0}^{9} \left[\sqrt{9-y} - \frac{1}{3}y \right] dy \\ &= \left[-\frac{2}{3}(9-y)^{\frac{3}{2}} - \frac{1}{6}y^2 \right]_{0}^{9} \\ &= \frac{9}{2}. \end{aligned}$$