MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 2.6 | **Math 2000 Worksheet** | WINTER 2020

SOLUTIONS

1. (a) First observe that

$$\int f(x,y)\,dx = \int \frac{y}{x^2+y^2}\,dx = y\cdot\frac{1}{y}\arctan\left(\frac{x}{y}\right) + g(y) = \arctan\left(\frac{x}{y}\right) + C(y).$$

To integrate with respect to y, we use u-substitution. We let $u = x^2+y^2$ so $\frac{1}{2}du = y\,dy$ and thus

$$\int f(x,y)\,dy = \int \frac{y}{x^2+y^2}\,dy = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C(x) = \frac{1}{2}\ln(x^2+y^2) + C(x).$$

(b) To integrate with respect to x, we use integration by parts, letting $w = \ln(x)$ so $dw = \frac{1}{x}\,dx$ and $dv = xy\,dx$ so $v = \frac{1}{2}x^2y$. Thus

$$\begin{aligned}\int f(x,y)\,dx &= \int xy\ln(x)\,dx = \frac{1}{2}x^2y\cdot\ln(x) - \int\left(\frac{1}{2}x^2y\cdot\frac{1}{x}\right)dx \\ &= \frac{1}{2}x^2y\ln(x) - \frac{1}{2}\int xy\,dx = \frac{1}{2}x^2y\ln(x) - \frac{1}{4}x^2y + C(y).\end{aligned}$$

Next,

$$\int f(x,y)\,dy = \int xy\ln(x)\,dy = \frac{1}{2}xy^2\ln(x) + C(x).$$

2. (a) We have

$$\begin{aligned}\int_0^{\pi}\int_0^4 y\cos\left(\frac{x}{4}\right)dy\,dx &= \int_0^{\pi}\left[\frac{1}{2}y^2\cos\left(\frac{x}{4}\right)\right]_{y=0}^{y=4}dx \\ &= \frac{1}{2}\int_0^{\pi}\left[16\cos\left(\frac{x}{4}\right)\right]dx \\ &= 8\int_0^{\pi}\cos\left(\frac{x}{4}\right)dx \\ &= 8\left[4\sin\left(\frac{x}{4}\right)\right]_0^{\pi} \\ &= 32\left[\sin\left(\frac{\pi}{4}\right) - \sin(0)\right] \\ &= 32\cdot\frac{\sqrt{2}}{2} \\ &= 16\sqrt{2}.\end{aligned}$$

(b) We have

$$\int_{-2}^{1}\int_{0}^{6}\frac{4y+3}{7-x}\,dx\,dy = \int_{-2}^{1}\Big[-(4y+3)\ln|7-x|\Big]_{0}^{6}\,dy$$
$$= \int_{-2}^{1} -(4y+3)[\ln(1)-\ln(7)]\,dy$$
$$= \ln(7)\int_{-2}^{1}(4y+3)\,dy$$
$$= \ln(7)\Big[2y^2+3y\Big]_{-2}^{1}$$
$$= 3\ln(7).$$

(c) We have

$$\int_{1}^{5}\int_{0}^{\frac{\pi}{4}} 3y^2\sec^2(x)\,dx\,dy = \int_{1}^{5}\Big[3y^2\tan(x)\Big]_{0}^{\frac{\pi}{4}}\,dy$$
$$= \int_{1}^{5} 3y^2\,dy$$
$$= \Big[y^3\Big]_{1}^{5}$$
$$= 124.$$

3. We can integrate in either order. If we integrate with respect to x first, then we write

$$\iint_R (x^2-xy+y^3-4)\,dA = \int_{-2}^{1}\int_{-1}^{2}(x^2-xy+y^3-4)\,dx\,dy$$
$$= \int_{-2}^{1}\left[\frac{1}{3}x^3-\frac{1}{2}x^2y+xy^3-4x\right]_{x=-1}^{x=2}\,dy$$
$$= \int_{-2}^{1}\left[\left(\frac{8}{3}-2y+2y^3-8\right)-\left(-\frac{1}{3}-\frac{1}{2}y-y^3+4\right)\right]\,dy$$
$$= \int_{-2}^{1}\left(-9-\frac{3}{2}y+3y^3\right)\,dy$$
$$= \left[-9y-\frac{3}{4}y^2+\frac{3}{4}y^4\right]_{-2}^{1}$$
$$= \left(-9-\frac{3}{4}+\frac{3}{4}\right)-(18-3+12)$$
$$= -36.$$

Alternatively, we could begin by integrating with respect to y:

$$\iint_R (x^2 - xy + y^3 - 4)\,dA = \int_{-1}^{2}\int_{-2}^{1} (x^2 - xy + y^3 - 4)\,dy\,dx$$
$$= \int_{-1}^{2} \left[x^2y - \frac{1}{2}xy^2 + \frac{1}{4}y^4 - 4y\right]_{y=-2}^{y=1} dx$$
$$= \int_{-1}^{2} \left[\left(x^2 - \frac{1}{2}x + \frac{1}{4} - 4\right) - (-2x^2 - 2x + 4 + 8)\right] dx$$
$$= \int_{-1}^{2} \left(3x^2 + \frac{3}{2}x - \frac{63}{4}\right) dx$$
$$= \left[x^3 + \frac{3}{4}x^2 - \frac{63}{4}x\right]_{-1}^{2}$$
$$= \left(8 + 3 - \frac{63}{2}\right) - \left(-1 + \frac{3}{4} + \frac{63}{4}\right)$$
$$= -36.$$

4. If we start by integrating with respect to x, we have

$$V = \iint_R \frac{x^2}{\sqrt{9-y^2}}\,dA$$
$$= \int_0^{\frac{3}{2}}\int_{-3}^{3} \frac{x^2}{\sqrt{9-y^2}}\,dx\,dy$$
$$= \int_0^{\frac{3}{2}} \left[\frac{1}{3}\cdot\frac{x^3}{\sqrt{9-y^2}}\right]_{x=-3}^{x=3} dy$$
$$= \int_0^{\frac{3}{2}} \left(\frac{9}{\sqrt{9-y^2}} + \frac{9}{\sqrt{9-y^2}}\right) dy$$
$$= 18\int_0^{\frac{3}{2}} \frac{1}{\sqrt{9-y^2}}\,dy$$
$$= 18\left[\arcsin\left(\frac{y}{3}\right)\right]_0^{\frac{3}{2}}$$
$$= 18\left[\arcsin\left(\frac{1}{2}\right) - \arcsin(0)\right]$$
$$= 18\cdot\frac{\pi}{6}$$
$$= 3\pi.$$