MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Section 2.5	**Math 2000 Worksheet**	Winter 2020

SOLUTIONS

1. (a) The first-order partial derivatives are

$$f_x(x,y) = 2x - y \quad \text{and} \quad f_y(x,y) = 2y - x - 3.$$

Setting the first of these equal to zero yields $y = 2x$. Substituting this into the second equation gives

$$3 + x - 2(2x) = 0 \quad \Longrightarrow \quad 3x = 3$$

and so $x = 1$ (and $y = 2$). Thus $(1,2)$ is the only critical point.
Next we have

$$f_{xx}(x,y) = 2, \quad f_{yy}(x,y) = 2, \quad f_{xy}(x,y) = -1$$

so the discriminant is

$$D = f_{xx}(1,2)f_{yy}(1,2) - [f_{xy}(1,2)]^2 = 4 - 1 = 3 > 0.$$

Since $f_{xx}(1,2) = 2 > 0$, then, $(1,2)$ is a relative minimum.

(b) This time the first-order partial derivatives are

$$f_x(x,y) = 3x^2 + 6y - 6 \quad \text{and} \quad f_y(x,y) = -12y + 6x.$$

If $f_y(x,y) = 0$ then $x = 2y$. Thus, setting $f_x(x,y) = 0$ and using this result, we have

$$3(2y)^2 + 6y - 6 = 0 \quad \Longrightarrow \quad 12y^2 + 6y - 6 = 0 \quad \Longrightarrow \quad 2y^2 + y - 1 = 0$$

so $(2y-1)(y+1) = 0$. Hence $y = \frac{1}{2}$ (so $x = 1$) or $y = -1$ (so $x = -2$). This provides us with two critical points: $\left(1, \frac{1}{2}\right)$ and $(-2,-1)$.
Next, the second-order partial derivatives are

$$f_{xx}(x,y) = 6x, \quad f_{yy}(x,y) = -12, \quad f_{xy}(x,y) = 6.$$

At $\left(1, \frac{1}{2}\right)$ the discriminant is

$$D = f_{xx}\left(1, \frac{1}{2}\right) f_{yy}\left(1, \frac{1}{2}\right) - \left[f_{xy}\left(1, \frac{1}{2}\right)\right]^2 = -72 - 36 = -100 < 0,$$

so this is a saddle point.
At $(-2,-1)$ the discriminant is

$$D = f_{xx}(-2,-1)f_{yy}(-2,-1) - [f_{xy}(-2,-1)]^2 = 144 - 36 = 108 > 0,$$

and since $f_{xx}(-2,-1) = -12 < 0$, this is a relative maximum.

(c) First we find the first-order partial derivatives:

$$f_x(x, y) = 6x^5 - 6y \quad \text{and} \quad f_y(x, y) = 6y^5 - 6x.$$

Setting the first of these equal to zero gives $6x^5 - 6y = 0$ so $y = x^5$ and so we obtain from the second equation $6(x^5)^5 - 6x = 6x^{25} - 6x = 6x(x^{24} - 1) = 0$ so $x = 0$ (and $y = 0$), $x = 1$ (and $y = 1$), or $x = -1$ (and $y = -1$), giving us three critical points: $(0, 0)$, $(1, 1)$ and $(-1, -1)$.

To determine the nature of each of these critical points, we require the second derivatives:

$$f_{xx}(x, y) = 30x^4, \quad f_{yy}(x, y) = 30y^4, \quad f_{xy}(x, y) = -6.$$

For $(0, 0)$ this gives

$$D = f_{xx}(0, 0)f_{yy}(0, 0) - [f_{xy}(0, 0)]^2 = 0 - 36 = -36 < 0,$$

so $(x, y) = (0, 0)$ is a saddle point. For $(1, 1)$ we have

$$D = f_{xx}(1, 1)f_{yy}(1, 1) - [f_{xy}(1, 1)]^2 = 900 - 36 = 864 > 0$$

and since $f_{xx}(1, 1) = 30 > 0$, $(x, y) = (1, 1)$ is a relative minimum. For $(-1, -1)$, we also get

$$D = f_{xx}(-1, -1)f_{yy}(-1, -1) - [f_{xy}(-1, -1)]^2 = 900 - 36 = 864 > 0$$

with $f_{xx}(-1, -1) = 30 > 0$, so $(x, y) = (-1, -1)$ is also a relative minimum.

(d) We again begin by computing the first-order partial derivatives:

$$f_x(x, y) = 3x^2y^2 - 36x \quad \text{and} \quad f_y(x, y) = 2x^3y - 54y.$$

Setting the latter equal to zero gives $2y(x^3 - 27) = 0$ and hence either $y = 0$ or $x = 3$. If $y = 0$, setting $f_x(x, y) = 0$ implies $-36x = 0$ so $x = 0$. By the same process, if $x = 3$, we get $27y^2 - 108 = 0$ so $y^2 = 4$ and either $y = 2$ or $y = -2$. Thus the three critical points are $(0, 0)$, $(3, 2)$ and $(3, -2)$.

The second derivatives are

$$f_{xx}(x, y) = 6xy^2 - 36, \quad f_{yy}(x, y) = 2x^3 - 54, \quad f_{xy}(x, y) = 6x^2y.$$

For $(0, 0)$ we have

$$D = f_{xx}(0, 0)f_{yy}(0, 0) - [f_{xy}(0, 0)]^2 = (-36)(-54) - 0 = 1944 > 0$$

and since $f_{xx}(x, y) = -36 < 0$, $(x, y) = (0, 0)$ is a relative maximum. For $(3, 2)$ we have

$$D = f_{xx}(3, 2)f_{yy}(3, 2) - [f_{xy}(3, 2)]^2 = (36)(0) - (108)^2 = -11664 < 0,$$

so $(x, y) = (3, 2)$ is a saddle point. For $(3, -2)$, we also get

$$D = f_{xx}(3, -2)f_{yy}(3, -2) - [f_{xy}(3, -2)]^2 = (36)(0) - (108)^2 = -11664 < 0,$$

so $(x, y) = (3, -2)$ is another saddle point.