MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 2.3 **Math 2000 Worksheet** WINTER 2020

SOLUTIONS

1. (a) $\dfrac{\partial z}{\partial x} = \cos(x)\cos(y) \qquad \dfrac{\partial z}{\partial y} = -\sin(x)\sin(y)$

   (b) $f_x(x, y) = y^x \ln(y) \qquad f_y(x, y) = xy^{x-1}$

   (c)

$$\frac{\partial f(s,t)}{\partial s} = \frac{1}{1+\left(\frac{s^2}{t^2}\right)^2}\left(\frac{2s}{t^2}\right) = \frac{2st^2}{s^4+t^4}$$
$$\frac{\partial f(s,t)}{\partial t} = \frac{1}{1+\left(\frac{s^2}{t^2}\right)^2}\left(-\frac{2s^2}{t^3}\right) = -\frac{2s^2t}{s^4+t^4}$$

   (d)

$$z_x(x,y) = -3\sin(3x-5y) \implies z_x\left(\frac{\pi}{6}, \frac{\pi}{15}\right) = -3\sin\left(\frac{\pi}{2}-\frac{\pi}{3}\right) = -3\sin\left(\frac{\pi}{6}\right) = -\frac{3}{2}$$
$$z_y(x,y) = 5\sin(3x-5y) \implies z_y\left(\frac{\pi}{6}, \frac{\pi}{15}\right) = 5\sin\left(\frac{\pi}{2}-\frac{\pi}{3}\right) = 5\sin\left(\frac{\pi}{6}\right) = \frac{5}{2}$$

   (e) $\dfrac{\partial w}{\partial x} = y^2z^3 \qquad \dfrac{\partial w}{\partial y} = 2xyz^3 \qquad \dfrac{\partial w}{\partial z} = 3xy^2z^2$

   (f)

$$f_x(x,y,z) = \frac{6y}{\sqrt{25-z^2}} \implies f_x(1,2,4) = 4$$
$$f_y(x,y,z) = \frac{6x}{\sqrt{25-z^2}} \implies f_y(1,2,4) = 2$$
$$f_z(x,y,z) = \frac{6xyz}{(25-z^2)^{\frac{3}{2}}} \implies f_z(1,2,4) = \frac{16}{9}$$

2. We have

$$\begin{array}{ll} f_x(x,y) = ye^y & f_y(x,y) = xe^y + xye^y \\ f_{xx}(x,y) = 0 & f_{xy}(x,y) = e^y + ye^y \\ f_{yx}(x,y) = e^y + ye^y & f_{yy}(x,y) = xe^y + xe^y + xye^y = 2xe^y + xye^y \end{array}$$

   Since $f_{xy}(x,y) = f_{yx}(x,y)$, Clairaut's Theorem is satisfied.

3. We have

$$\begin{array}{ll} z_x = 3e^{3x}\sin(5y) & z_y = 5e^{3x}\cos(5y) \\ z_{xx} = 9e^{3x}\sin(5y) & z_{xxy} = 45e^{3x}\cos(5y) \\ z_{yx} = 15e^{3x}\cos(5y) & z_{yxx} = 45e^{3x}\cos(5y) \\ & z_{yxy} = -75e^{3x}\sin(5y) \\ z_{yy} = -25e^{3x}\sin(5y) & z_{yyx} = -75e^{3x}\sin(5y). \end{array}$$

These results are consistent with Clairault's Theorem, because as long as we differentiate the same number of times by the same variables, the partial derivatives are equal; that is, $z_{xxy} = z_{yxx}$ and $z_{yxy} = z_{yyx}$.

4. (a) We have

$$\begin{array}{ll} f_x(x,y) = 2x & f_y(x,y) = -2y \\ f_{xx}(x,y) = 2 & f_{yy}(x,y) = -2 \end{array}$$

so

$$f_{xx}(x,y) + f_{yy}(x,y) = 2 - 2 = 0,$$

and so this is a solution to Laplace's equation.

(b) We have

$$\begin{array}{ll} f_x(x,y) = 2x & f_y(x,y) = 2y \\ f_{xx}(x,y) = 2 & f_{yy}(x,y) = 2 \end{array}$$

so

$$f_{xx}(x,y) + f_{yy}(x,y) = 2 + 2 = 4 \neq 0,$$

and hence this is not a solution to Laplace's equation.

(c) We have

$$\begin{array}{ll} f_x(x,y) = \dfrac{4x}{x^2+y^2} & f_y(x,y) = \dfrac{4y}{x^2+y^2} \\ f_{xx}(x,y) = \dfrac{4y^2-4x^2}{(x^2+y^2)^2} & f_{yy}(x,y) = \dfrac{4x^2-4y^2}{(x^2+y^2)^2} \end{array}$$

so

$$f_{xx}(x,y) + f_{yy}(x,y) = \frac{4y^2-4x^2}{(x^2+y^2)^2} + \frac{4x^2-4y^2}{(x^2+y^2)^2} = 0,$$

and so this is a solution to Laplace's equation.

(d) We have

$$f_x(x,y) = -e^{-x}\cos(y) + e^{-y}\sin(x)$$
$$f_y(x,y) = -e^{-x}\sin(y) + e^{-y}\cos(x)$$
$$f_{xx}(x,y) = e^{-x}\cos(y) + e^{-y}\cos(x)$$
$$f_{yy}(x,y) = -e^{-x}\cos(y) - e^{-y}\cos(x)$$

so

$$f_{xx}(x,y) + f_{yy}(x,y) = e^{-x}\cos(y) + e^{-y}\cos(x) - e^{-x}\cos(y) - e^{-y}\cos(x) = 0,$$

and so this is a solution to Laplace's equation.

5. We have

$$\frac{\partial u}{\partial x} = k\cos(kx)\sin(\alpha kt) \qquad \frac{\partial u}{\partial t} = \alpha k\sin(kx)\cos(\alpha kt)$$
$$\frac{\partial^2 u}{\partial x^2} = -k^2\sin(kx)\sin(\alpha kt) \qquad \frac{\partial^2 u}{\partial t^2} = -\alpha^2 k^2\sin(kx)\sin(\alpha kt)$$

so

$$\frac{\partial^2 u}{\partial t^2} = -\alpha^2 k^2\sin(kx)\sin(\alpha kt) = \alpha^2\frac{\partial^2 u}{\partial x^2},$$

as desired.

6. We have

$$f_x(x,y) = e^y + ye^x \qquad f_y(x,y) = xe^y + e^x \qquad f_{xx}(x,y) = ye^x$$
$$f_{xy}(x,y) = e^y + e^x \qquad f_{yy}(x,y) = xe^y \qquad f_{xxx}(x,y) = ye^x$$
$$f_{xxy}(x,y) = e^x \qquad f_{xyy}(x,y) = e^y \qquad f_{yyy}(x,y) = xe^y$$

so

$$f_{xxx}(x,y) + f_{yyy}(x,y) = ye^x + xe^y = yf_{xxy}(x,y) + xf_{xyy}(x,y)$$

as desired.