MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

TEST 1 **MATHEMATICS 2000** WINTER 2020

SOLUTIONS

[4] 1. (a) We have

$$\lim_{i\to\infty} a_i = \lim_{i\to\infty} \frac{2i^3(i-5)}{(3i^2-4)(3i^2+4)} = \lim_{i\to\infty} \frac{2i^4-10i^3}{9i^4-16} \cdot \frac{\frac{1}{i^4}}{\frac{1}{i^4}}$$
$$= \lim_{i\to\infty} \frac{2-\frac{10}{i}}{9-\frac{16}{i^4}}$$
$$= \frac{2-0}{9-0}$$
$$= \frac{2}{9}.$$

[4] (b) [WORKSHEET 1.2, #2(d)] We use the Squeeze Theorem. Observe that since $0 \leq \sin^2(i) \leq 1$ for all i,

$$0 \leq \frac{\sin^2(i)}{5^i} \leq \frac{1}{5^i}.$$

But

$$\lim_{i\to\infty} 0 = 0 \quad \text{and} \quad \lim_{i\to\infty} \frac{1}{5^i} = \lim_{i\to\infty} \left(\frac{1}{5}\right)^i = 0,$$

so by the Squeeze Theorem,

$$\lim_{i\to\infty} \frac{\sin^2(i)}{5^i} = 0$$

and hence $\{a_i\}$ converges to 0.

[5] (c) The corresponding function is

$$f(x) = \frac{\ln(\ln(x))}{\sqrt{x}}$$

and since $\lim_{x\to\infty} f(x)$ yields a $\frac{\infty}{\infty}$ indeterminate form, we apply l'Hôpital's Rule:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln(\ln(x))}{\sqrt{x}} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\frac{1}{\ln(x)} \cdot \frac{1}{x}}{\frac{1}{2\sqrt{x}}}$$
$$= \lim_{x\to\infty} \frac{2}{\sqrt{x}\ln(x)}$$
$$= 0.$$

Hence, by the Evaluation Theorem, $\lim_{i\to\infty} a_i = 0$ as well.

[3] 2. In #1(a) we showed that $\lim_{i\to\infty} a_i$ is non-zero. Hence, by the Divergence Test, the corresponding series $\sum_{i=1}^{\infty} a_i$ must be divergent.

[6] 3. First we will determine whether the sequence is monotonic. Since it contains factorial and factorial-like expressions, we cannot use the derivative of the corresponding function for this purpose. Instead, observe that

$$a_i = \frac{4(i-1)!}{1\cdot 5\cdot 9\cdots(4i-3)} \qquad \Longrightarrow \qquad a_{i+1} = \frac{4(i!)}{1\cdot 5\cdot 9\cdots(4i-3)(4i+1)}$$

so

$$\frac{a_{i+1}}{a_i} = \frac{4(i!)}{1\cdot 5\cdot 9\cdots(4i-3)(4i+1)}\cdot\frac{1\cdot 5\cdot 9\cdots(4i-3)}{4(i-1)!} = \frac{i}{4i+1}.$$

Since $4i+1 > i$ for all $i \geq 1$, we see that $\frac{a_{i+1}}{a_i} < 1$ and therefore $\{a_i\}$ is (monotonic) decreasing.

This also means that

$$a_1 = \frac{4\cdot 0!}{1} = 4$$

is an upper bound of the sequence. Furthermore, because the numerator and denominator both consist of products of positive integers, they must be positive and therefore $a_i > 0$ for all $i \geq 1$. Thus 0 is a lower bound of the sequence. Hence the sequence is bounded.

By the Bounded Monotonic Sequence Theorem, then, $\{a_i\}$ must converge.

[4] 4. [Worksheet 1.3, #5(b)] Rewriting gives

$$\sum_{i=0}^{\infty}\frac{3^i-4^i}{3^i4^i} = \sum_{i=0}^{\infty}\frac{3^i}{3^i4^i} - \sum_{i=0}^{\infty}\frac{4^i}{3^i4^i} = \sum_{i=0}^{\infty}\left(\frac{1}{4}\right)^i - \sum_{i=0}^{\infty}\left(\frac{1}{3}\right)^i.$$

Both of these series are geometric; the first has $r = \frac{1}{4}$ and the second has $r = \frac{1}{3}$. So then

$$\sum_{i=0}^{\infty}\frac{3^i-4^i}{3^i4^i} = \frac{1}{1-\frac{1}{4}} - \frac{1}{1-\frac{1}{3}} = \frac{4}{3} - \frac{3}{2} = -\frac{1}{6}.$$

[4] 5. First, the square root in the numerator will be defined only if $y - x \geq 0$, so we require $y \geq x$. Next, we must have a non-zero denominator, so $(x+3) \neq 0$ and hence $x \neq -3$. Thus the domain of $f(x, y)$ consists of all the points on and above the line $y = x$, excluding those on the vertical line $x = -3$.

[5] 6. [Worksheet 2.2, #1(e)] First we let $(x, y) \to (0, 0)$ along the line $y = 0$ so the limit becomes

$$\lim_{(x,y)\to(0,0)}\frac{12x^4y}{x^6+3y^3} = \lim_{x\to 0}\frac{0}{x^6} = \lim_{x\to 0} 0 = 0.$$

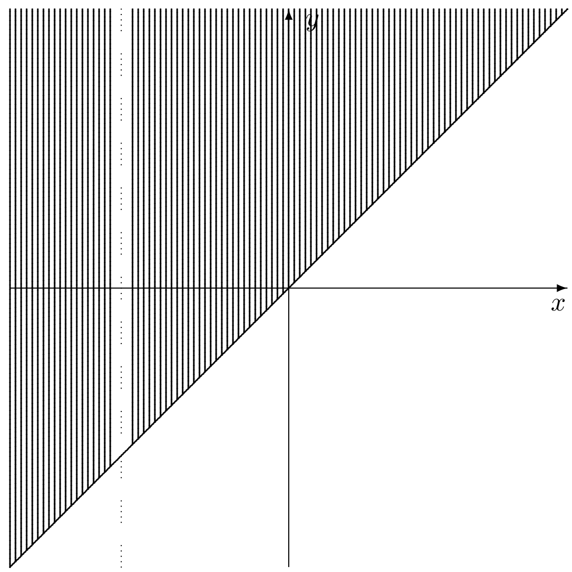

Figure 1: The domain of $f(x, y)$ for Question 5.

Next we could try letting $(x, y) \to (0, 0)$ along the line $x = 0$, but then we obtain

$$\lim_{(x,y)\to(0,0)} \frac{12x^4y}{x^6 + 3y^3} = \lim_{y\to 0} \frac{0}{3y^3} = \lim_{x\to 0} 0 = 0,$$

which is the same value we have already computed. Similarly, if we use the path $y = x$, we get

$$\lim_{(x,y)\to(0,0)} \frac{12x^4y}{x^6 + 3y^3} = \lim_{x\to 0} \frac{12x^4}{x^6 + 3x^3} = \lim_{x\to 0} \frac{12x}{x^3 + 3} = 0.$$

But if we consider the path $y = x^2$, we have

$$\lim_{(x,y)\to(0,0)} \frac{12x^4y}{x^6 + 3y^3} = \lim_{x\to 0} \frac{12x^6}{4x^6} = \lim_{x\to 0} 3 = 3,$$

and since this differs from the previous values, we conclude that the limit does not exist.

[5] 7. We have

$$\begin{aligned}
z_x &= \left[1 \cdot \ln(x) + x \cdot \frac{1}{x}\right] \sin(5y) \\
&= \ln(x)\sin(5y) + \sin(5y), \\
z_y &= x\ln(x) \cdot \cos(5y) \cdot 5 \\
&= 5x\ln(x)\cos(5y), \\
z_{xx} &= \frac{1}{x} \cdot \sin(5y) + 0 \\
&= \frac{\sin(5y)}{x}, \\
z_{yy} &= 5x\ln(x) \cdot [-\sin(5y)] \cdot 5 \\
&= -25x\ln(x)\sin(5y), \\
z_{xy} = z_{yx} &= \ln(x) \cdot \cos(5y) \cdot 5 + \cos(5y) \cdot 5 \\
&= 5\ln(x)\cos(5y) + 5\cos(5y).
\end{aligned}$$