MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 1.9 **Math 2000 Worksheet** WINTER 2020

SOLUTIONS

1. (a) First we find the radius of convergence; we use the Ratio Test with

$$k_i = \frac{(-1)^i}{i+1} \quad \text{and} \quad k_{i+1} = \frac{(-1)^{i+1}}{i+2}$$

so then

$$\lim_{i\to\infty}\left|\frac{k_{i+1}}{k_i}\right| = \lim_{i\to\infty}\frac{i+1}{i+2} = 1 = \rho$$

so the radius of convergence is $R = \frac{1}{\rho} = 1$. Hence the series converges for all x such that $|x-2| < 1$, that is, for $-1 < x-2 < 1$ or $1 < x < 3$. Now we check the endpoints. When $x = 3$, the given series becomes

$$\sum_{i=0}^{\infty}\frac{(-1)^i}{i+1}$$

which is convergent by the Alternating Series Test. When $x = 1$, the given series becomes

$$\sum_{i=0}^{\infty}\frac{(-1)^i}{i+1}(-1)^i = \sum_{i=0}^{\infty}\frac{(-1)^{2i}}{i+1} = \sum_{i=0}^{\infty}\frac{1}{i+1}$$

which is divergent by Limit Comparison with the harmonic series. So the interval of convergence is $(1,3]$.

(b) The given series is

$$f(x) = \sum_{i=0}^{\infty}\frac{(-1)^i}{i+1}(x-2)^i = \frac{1}{1} - \frac{1}{2}(x-2) + \frac{1}{3}(x-2)^2 - \frac{1}{4}(x-2)^3 + \frac{1}{5}(x-2)^4 - \cdots,$$

so differentiating it yields

$$f'(x) = \sum_{i=1}^{\infty}\frac{(-1)^i i}{i+1}(x-2)^{i-1} = -\frac{1}{2} + \frac{2}{3}(x-2) - \frac{3}{4}(x-2)^2 + \frac{4}{5}(x-2)^3 - \cdots.$$

The radius of convergence is the same as in part (a), namely, $R = 1$ so this series also converges for $1 < x < 3$ and we again need to check the endpoints. When $x = 3$, the differentiated series becomes

$$\sum_{i=1}^{\infty}\frac{(-1)^i i}{i+1}$$

which diverges by the Divergence Test. When $x = 1$, it becomes

$$\sum_{i=1}^{\infty}\frac{(-1)^i i}{i+1}(-1)^{i-1} = \sum_{i=1}^{\infty}\frac{-i}{i+1}$$

which also diverges by the Divergence Test. So the interval of convergence this time is $(1,3)$.

(c) Integrating the given series

$$f(x) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i+1}(x-2)^i = \frac{1}{1} - \frac{1}{2}(x-2) + \frac{1}{3}(x-2)^2 - \frac{1}{4}(x-2)^3 + \frac{1}{5}(x-2)^4 - \cdots$$

gives

$$\begin{aligned}\int f(x)\,dx &= C + \sum_{i=0}^{\infty} \frac{(-1)^i}{(i+1)^2}(x-2)^{i+1} \\ &= C + \frac{1}{1}(x-2) - \frac{1}{2^2}(x-2)^2 + \frac{1}{3^2}(x-2)^3 - \frac{1}{4^2}(x-2)^4 + \cdots,\end{aligned}$$

for some constant C which will not affect the interval of convergence. Again, the radius of convergence must be $R = 1$ giving convergence for $1 < x < 3$, and so we check the endpoints. For $x = 3$, the integrated series becomes

$$\sum_{i=0}^{\infty} \frac{(-1)^i}{(i+1)^2}$$

which converges by the Alternating Series Test. For $x = 1$, it becomes

$$\sum_{i=0}^{\infty} \frac{(-1)^i}{(i+1)^2}(-1)^{i+1} = \sum_{i=0}^{\infty} \frac{-1}{(i+1)^2}$$

which converges by comparison (Limit or Direct) with the convergent p-series $\displaystyle\sum_{i=0}^{\infty} \frac{1}{i^2}$.

Hence the interval of convergence is $[1, 3]$.

2. (a) We write

$$\frac{8}{4x+7} = \frac{\frac{8}{7}}{1+\frac{4}{7}x} = \frac{8}{7}\sum_{i=0}^{\infty}\left(-\frac{4}{7}x\right)^i = \sum_{i=0}^{\infty}\frac{8}{7}\left(-\frac{4}{7}\right)^i x^i = \sum_{i=0}^{\infty} 2(-1)^i\left(\frac{4}{7}\right)^{i+1} x^i,$$

which will converge for all $\left|-\frac{4}{7}x\right| < 1$, that is, for $-1 < \frac{4}{7}x < 1$ or $-\frac{7}{4} < x < \frac{7}{4}$.

(b) Observe that if we set $f(x) = \frac{1}{1-x}$ then

$$f'(x) = \frac{1}{(1-x)^2} \quad \text{and} \quad f''(x) = \frac{2}{(1-x)^3}.$$

So we write

$$\frac{2}{(1-x)^3} = \frac{d^2}{dx^2}\left[\sum_{i=0}^{\infty} x^i\right] = \frac{d}{dx}\left[\sum_{i=1}^{\infty} i x^{i-1}\right] = \sum_{i=2}^{\infty} i(i-1)x^{i-2}.$$

This certainly converges for $|x| < 1$, but because we have differentiated, we must check the endpoints. At $x = 1$, the differentiated series becomes

$$\sum_{i=2}^{\infty} i(i-1)$$

which diverges by the Divergence Test. At $x = -1$, the differentiated series becomes

$$\sum_{i=2}^{\infty}(-1)^{i-2}i(i-1)$$

which also diverges by the Divergence Test. So the interval of convergence remains $(-1, 1)$.

(c) Note first that

$$\frac{d}{dx}[\ln(5x+1)] = \frac{5}{5x+1}$$

$$\ln(5x+1) = 5\int \frac{dx}{1+5x} = 5\int \left[\sum_{i=0}^{\infty}(-5x)^i\right] = 5\int \left[\sum_{i=0}^{\infty}(-5)^i x^i\right]$$
$$= 5\left[C + \sum_{i=0}^{\infty}\frac{(-5)^i}{i+1}x^{i+1}\right] = C + \sum_{i=0}^{\infty}\frac{(-1)^i 5^{i+1}}{i+1}x^{i+1}.$$

To solve for the constant C, we observe that when $x = 0$, $\ln(5x+1) = \ln(1) = 0$. Substituting this into the series, we see that $C = 0$ as well. Thus

$$\ln(5x+1) = \sum_{i=0}^{\infty}\frac{(-1)^i 5^{i+1}}{i+1}x^{i+1}.$$

We are guaranteed convergence for $|-5x| < 1$, that is, for $-1 < 5x < 1$ or $-\frac{1}{5} < x < \frac{1}{5}$. We check the endpoints. For $x = \frac{1}{5}$ the integrated series becomes

$$\sum_{i=0}^{\infty}\frac{(-1)^i}{i+1}$$

which converges by the Alternating Series Test. For $x = -\frac{1}{5}$, it becomes

$$\sum_{i=0}^{\infty}\frac{-1}{i+1}$$

which diverges (try Limit Comparison with the harmonic series). So the interval of convergence is $\left(-\frac{1}{5}, \frac{1}{5}\right]$.

(d) Observe that

$$\int \frac{-4x^3}{(1+x^4)^2}\,dx = \frac{1}{1+x^4} + C.$$

So then

$$\frac{-4x^3}{(1-x^4)^2} = \frac{d}{dx}\left[\frac{1}{1+x^4}\right] = \frac{d}{dx}\left[\sum_{i=0}^{\infty}(-x^4)^i\right] = \frac{d}{dx}\left[\sum_{i=0}^{\infty}(-1)^i x^{4i}\right] = \sum_{i=1}^{\infty}(-1)^i 4ix^{4i-1}.$$

This converges for $|-x^4| < 1$, that is, for $-1 < x < 1$. As usual, we check the endpoints. At $x = 1$, the differentiated series becomes

$$\sum_{i=1}^{\infty} (-1)^i 4i$$

which diverges by the Divergence Test. At $x = -1$, the differentiated series becomes

$$\sum_{i=1}^{\infty} (-1)^{i+1} 4i$$

which also diverges by the Divergence Test. Hence the interval of convergence is still $(-1, 1)$.