MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 1.8 **Math 2000 Worksheet** WINTER 2020

SOLUTIONS

1. (a) We use the Ratio Test with

$$k_i = \frac{1}{2i+1} \quad \text{so} \quad k_{i+1} = \frac{1}{2i+3}.$$

Then

$$\lim_{i\to\infty}\left|\frac{c_{i+1}}{c_i}\right| = \lim_{i\to\infty}\frac{2i+1}{2i+3} = 1 = \rho.$$

Hence the radius of convergence is $R = \frac{1}{\rho} = 1$, and so the power series converges for $|x| < 1$, that is, for $-1 < x < 1$. We check the endpoints $x = \pm 1$. For $x = 1$, the series becomes $\displaystyle\sum_{i=0}^{\infty}\frac{1}{2i+1}$ which diverges (try the Limit Comparison Test with the harmonic series). For $x = -1$, the series becomes $\displaystyle\sum_{i=0}^{\infty}\frac{(-1)^i}{2i+1}$ which converges by the Alternating Series Test. Hence the interval of convergence is $[-1, 1)$.

(b) We use the Root Test with $k_i = \dfrac{1}{i^i}$. Then

$$\lim_{i\to\infty}|k_i|^{\frac{1}{i}} = \lim_{i\to\infty}\frac{1}{i} = 0 = \rho.$$

So the radius of convergence is $R = \infty$ and the interval of convergence must be $\mathbb{R}$.

(c) We use the Ratio Test with

$$k_i = \frac{1}{3i(i+1)} \quad \text{so} \quad k_{i+1} = \frac{1}{3(i+1)(i+2)}.$$

Then

$$\lim_{i\to\infty}\left|\frac{k_{i+1}}{k_i}\right| = \lim_{i\to\infty}\frac{3i(i+1)}{3(i+1)(i+2)} = 1 = \rho.$$

So the radius of convergence is $R = \frac{1}{\rho} = 1$ and the series converges for all x such that $|x-4| < 1$, that is, for $-1 < x-4 < 1$ or $3 < x < 5$. We check the endpoints $x = 3$ and $x = 5$. For $x = 5$ the series becomes $\displaystyle\sum_{i=0}^{\infty}\frac{1}{3i(i+1)}$ which converges (try the Limit Comparison Test with the convergent p-series $\displaystyle\sum_{i=0}^{\infty}\frac{1}{i^2}$). For $x = 3$, the series becomes $\displaystyle\sum_{i=0}^{\infty}\frac{(-1)^i}{3i(i+1)}$ which converges by the Alternating Series Test. Hence the interval of convergence is $[3, 5]$.

(d) We use the Ratio Test with

$$k_i = \frac{i}{(i^2+1)4^i} \quad \text{so} \quad k_{i+1} = \frac{i+1}{((i+1)^2+1)4^{i+1}}.$$

Then

$$\lim_{i\to\infty}\left|\frac{k_{i+1}}{k_i}\right| = \lim_{i\to\infty}\frac{i+1}{((i+1)^2+1)4^{i+1}}\cdot\frac{(i^2+1)4^i}{i} = \lim_{i\to\infty}\frac{(i+1)(i^2+1)}{4i[(i+1)^2+1]} = \frac{1}{4} = \rho.$$

Then the radius of convergence is $R = \frac{1}{\rho} = 4$ and the series converges for all x such that $|x+7| < 4$, that is, for $-4 < x+7 < 4$ or $-11 < x < -3$. We check the endpoints $x = -11$ and $x = -3$. For $x = -3$, the series becomes

$$\sum_{i=0}^{\infty}\frac{i}{(i^2+1)4^i}4^i = \sum_{i=0}^{\infty}\frac{i}{(i^2+1)}$$

which diverges (try the Limit Comparison Test with the harmonic series). For $x = -11$, the series becomes

$$\sum_{i=0}^{\infty}\frac{i}{(i^2+1)4^i}(-4)^i = \sum_{i=0}^{\infty}(-1)^i\frac{i}{(i^2+1)}$$

which converges by the Alternating Series Test. Hence the interval of convergence is $[-11,-3)$.

(e) Note that the starting index is $i = 2$, but this will affect only the sum of the power series (were we able to find it), not its convergence properties. We use the Ratio Test with

$$k_i = \ln(i) \quad \text{so} \quad k_{i+1} = \ln(i+1).$$

Then

$$\lim_{i\to\infty}\left|\frac{k_{i+1}}{k_i}\right| = \lim_{i\to\infty}\frac{\ln(i+1)}{\ln(i)}.$$

This is an $\frac{\infty}{\infty}$ indeterminate form so we let $f(x) = \frac{\ln(x+1)}{\ln(x)}$ and use L'Hospital Rule:

$$\lim_{x\to\infty}\frac{\ln(x+1)}{\ln(x)} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{\frac{1}{x+1}}{\frac{1}{x}} = \lim_{x\to\infty}\frac{x}{x+1} = 1 = \rho.$$

Hence the radius of convergence is $R = \frac{1}{\rho} = 1$ and the series converges for all $|x| < 1$, that is, for $-1 < x < 1$. We check the endpoints $x = \pm 1$. For $x = 1$, the series becomes $\sum_{i=2}^{\infty}\ln(i)$ which diverges by the Divergence Test. For $x = -1$, the series becomes $\sum_{i=2}^{\infty}(-1)^i\ln(i)$ which diverges for the same reason. So the interval of convergence is $(-1,1)$.

(f) We use the Ratio Test with

$$k_i = \frac{(-1)^i (2i)!}{i!} \quad \text{so} \quad k_{i+1} = \frac{(-1)^{i+1}(2i+2)!}{(i+1)!}.$$

Then

$$\lim_{i\to\infty} \left| \frac{k_{i+1}}{k_i} \right| = \lim_{i\to\infty} \frac{(2i+2)!}{(i+1)!} \cdot \frac{i!}{(2i)!} = \lim_{i\to\infty} \frac{(2i+1)(2i+2)}{i+1} = \infty = \rho.$$

So the radius of convergence is $R = 0$ and therefore the interval of convergence consists of only the centre of the power series, $x = 12$.

(g) First we need to write the series as

$$\sum_{i=1}^{\infty} \frac{2\cdot 4\cdot 6\cdots(2i)}{1\cdot 3\cdot 5\cdots(2i-1)}(5x-1)^i = \sum_{i=1}^{\infty} \frac{2\cdot 4\cdot 6\cdots(2i)}{1\cdot 3\cdot 5\cdots(2i-1)} \left[5\left(x-\frac{1}{5}\right)\right]^i$$
$$= \sum_{i=1}^{\infty} \frac{5^i[2\cdot 4\cdot 6\cdots(2i)]}{1\cdot 3\cdot 5\cdots(2i-1)} \left(x-\frac{1}{5}\right)^i.$$

Now we can use the Ratio Test with

$$k_i = \frac{5^i[2\cdot 4\cdot 6\cdots(2i)]}{1\cdot 3\cdot 5\cdots(2i-1)} \quad \text{so} \quad k_{i+1} = \frac{5^{i+1}[2\cdot 4\cdot 6\cdots(2i)\cdot(2i+2)]}{1\cdot 3\cdot 5\cdots(2i-1)\cdot(2i+1)}.$$

Then

$$\lim_{i\to\infty} \left| \frac{k_{i+1}}{k_i} \right| = \lim_{i\to\infty} \frac{5^{i+1}[2\cdot 4\cdot 6\cdots(2i)\cdot(2i+2)]}{1\cdot 3\cdot 5\cdots(2i-1)\cdot(2i+1)} \cdot \frac{1\cdot 3\cdot 5\cdots(2i-1)}{5^i[2\cdot 4\cdot 6\cdots(2i)]}$$
$$= \lim_{i\to\infty} \frac{5(2i+2)}{2i+1} = 5 = \rho.$$

So $R = \frac{1}{\rho} = \frac{1}{5}$. The series converges for all x such that $\left|x - \frac{1}{5}\right| < \frac{1}{5}$, that is, for $0 < x < \frac{2}{5}$. We check the endpoints $x = 0$ and $x = \frac{2}{5}$. For $x = \frac{2}{5}$, the series becomes

$$\sum_{i=0}^{\infty} \frac{2\cdot 4\cdot 6\cdots(2i)}{1\cdot 3\cdot 5\cdots(2i-1)}.$$

Note that the factors in the numerator are all larger than the factors in the denominator and so

$$\lim_{i\to\infty} \frac{2\cdot 4\cdot 6\cdots(2i)}{1\cdot 3\cdot 5\cdots(2i-1)} = \infty.$$

Thus the series diverges by the Divergence Test. Similarly, the series obtained for $x = 0$,

$$\sum_{i=0}^{\infty} (-1)^i \frac{2\cdot 4\cdot 6\cdots(2i)}{1\cdot 3\cdot 5\cdots(2i-1)},$$

also diverges by the Divergence Test and thus the interval of convergence is $\left(0, \frac{2}{5}\right)$.

(h) We must use the Ratio Test from first principles, given the power of $2i$. We have

$$a_i = \frac{5^{2i+1}}{9^i}(x-3)^{2i} \quad \text{and} \quad a_{i+1} = \frac{5^{2i+3}}{9^{i+1}}(x-3)^{2i+2}$$

so

$$\begin{aligned} L &= \lim_{i\to\infty} \left| \frac{a_{i+1}}{a_i} \right| \\ &= \lim_{i\to\infty} \left| \frac{5^{2i+3}}{9^{i+1}}(x-3)^{2i+2} \cdot \frac{9^i}{5^{2i+1}(x-3)^{2i}} \right| \\ &= \lim_{i\to\infty} \frac{25}{9}(x-3)^2 \\ &= \frac{25}{9}(x-3)^2. \end{aligned}$$

The power series will converge if

$$\frac{25}{9}(x-3)^2 < 1 \quad \Longrightarrow \quad (x-3)^2 < \frac{9}{25} \quad \Longrightarrow \quad -\frac{3}{5} < x-3 < \frac{3}{5} \quad \Longrightarrow \quad \frac{12}{5} < x < \frac{18}{5}.$$

Hence the radius of convergence is $R = \frac{3}{5}$. At $x = \frac{18}{5}$, the power series becomes

$$\sum_{i=0}^{\infty} \frac{5^{2i+1}}{9^i} \left(\frac{3}{5}\right)^{2i} = \sum_{i=0}^{\infty} \frac{5^{2i+1}}{9^i} \left(\frac{9}{25}\right)^{i} = \sum_{i=0}^{\infty} 5,$$

which diverges by the Divergence Test. Similarly, at $x = \frac{12}{5}$, the power series becomes

$$\sum_{i=0}^{\infty} \frac{5^{2i+1}}{9^i} \left(-\frac{3}{5}\right)^{2i} = \sum_{i=0}^{\infty} \frac{5^{2i+1}}{9^i} \left(\frac{9}{25}\right)^{i} = \sum_{i=0}^{\infty} 5,$$

so it also diverges. Hence the interval of convergence is $\left(\frac{12}{5}, \frac{18}{5}\right)$.