

MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 1.6

Math 2000 Worksheet

WINTER 2020

SOLUTIONS

1. (a) The absolute series is $\sum_{i=2}^{\infty} \frac{\ln(i)}{i}$. On Worksheet 1.4, we used the Integral Test to show that this series diverges. Otherwise, we could also deduce this by trying the Limit Comparison Test with the (divergent) harmonic series $\sum_{i=2}^{\infty} \frac{1}{i}$:

$$\lim_{i \rightarrow \infty} \frac{\frac{\ln(i)}{i}}{\frac{1}{i}} = \lim_{i \rightarrow \infty} \ln(i) = \infty.$$

Hence the given series is not absolutely convergent. To see if it's conditionally convergent, we use the Alternating Series Test. Observe that

$$\lim_{i \rightarrow \infty} \frac{\ln(i)}{i} \stackrel{H}{=} \lim_{i \rightarrow \infty} \frac{\frac{1}{i}}{1} = \lim_{i \rightarrow \infty} \frac{1}{i} = 0,$$

as required. Also, letting $f(x) = \frac{\ln(x)}{x}$, we have

$$f'(x) = \frac{1 - \ln(x)}{x^2} < 0 \quad \text{for } x > 2,$$

so $\left\{ \frac{\ln(i)}{i} \right\}$ is decreasing. Thus, by the Alternating Series Test, the given series is convergent. Since it is convergent but not absolutely convergent, we conclude that the given series is conditionally convergent.

- (b) The absolute series is $\sum_{i=1}^{\infty} \frac{1}{3i^2 + 1}$. To test its convergence, we use the Limit Comparison

Test with the convergent p -series $\sum_{i=1}^{\infty} \frac{1}{i^2}$:

$$\lim_{i \rightarrow \infty} \frac{\frac{1}{3i^2+1}}{\frac{1}{i^2}} = \lim_{i \rightarrow \infty} \frac{i^2}{3i^2 + 1} = \frac{1}{3},$$

so the absolute series is convergent. Hence the given series must also converge, and it is absolutely convergent.

- (c) Observe that

$$\lim_{i \rightarrow \infty} \frac{\sqrt{i}}{1 + 4\sqrt{i}} = \frac{1}{4},$$

so by the Divergence Test, the given series is divergent. (Note that this means its absolute series diverges also.)

- (d) The absolute series is $\sum_{i=1}^{\infty} \frac{\sqrt[3]{i}}{i} = \sum_{i=1}^{\infty} \frac{1}{i^{\frac{2}{3}}}$, which is a divergent p -series. So the given series is not absolutely convergent. We use the Alternating Series Test on the given series. We observe that

$$\lim_{i \rightarrow \infty} \frac{1}{i^{\frac{2}{3}}} = 0.$$

Also, setting $f(x) = \frac{1}{x^{\frac{2}{3}}} = x^{-\frac{2}{3}}$, we have

$$f'(x) = -\frac{2}{3}x^{-\frac{5}{3}} < 0,$$

so $\left\{ \frac{1}{i^{\frac{2}{3}}} \right\}$ is decreasing. Hence, by the Alternating Series Test, the given series is convergent, and so it is conditionally convergent.

- (e) The absolute series is $\sum_{i=1}^{\infty} \frac{1}{e^{i^3}}$. To determine its convergence, we use the Direct Comparison Test with the convergent geometric series $\sum_{i=1}^{\infty} \left(\frac{1}{e}\right)^i$, observing that for $i \geq 1$,

$$\begin{aligned} (e^i)^3 &> e^i \\ \frac{1}{(e^i)^3} &< \frac{1}{e^i} \\ \frac{1}{e^{i^3}} &< \left(\frac{1}{e}\right)^i. \end{aligned}$$

Hence the absolute series convergent, and therefore the given series is absolutely convergent.

- (f) First note that $\cos(i\pi) = (-1)^i$ so the series can be written $\sum_{i=1}^{\infty} \frac{(-1)^i}{i^{\frac{1}{4}}}$ and the absolute series is $\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{4}}}$. This is a divergent p -series, so the given series is not absolutely convergent. To test the convergence of the given series, we use the Alternating Series Test. Note that

$$\lim_{i \rightarrow \infty} \frac{1}{i^{\frac{1}{4}}} = 0.$$

Also, letting $f(x) = \frac{1}{x^{\frac{1}{4}}} = x^{-\frac{1}{4}}$, we have

$$f'(x) = -\frac{1}{4}x^{-\frac{5}{4}} < 0$$

so $\left\{ \frac{1}{i^{\frac{1}{4}}} \right\}$ is decreasing. Thus, by the Alternating Series Test, the given series is convergent and hence conditionally convergent.

2. Let $a_i = \frac{1}{4^i i!}$. First note that

$$\lim_{i \rightarrow \infty} \frac{1}{4^i i!} = 0.$$

To check that $\{a_i\}$ is decreasing, note that $a_{i+1} = \frac{1}{4^{i+1}(i+1)!}$ so

$$\frac{a_{i+1}}{a_i} = \frac{1}{4^{i+1}(i+1)!} \cdot \frac{4^i i!}{1} = \frac{4^i}{4^{i+1}} \cdot \frac{i!}{(i+1)!} = \frac{1}{4(i+1)} < 1 \quad \text{for } i \geq 1.$$

Hence we can use the remainder estimate for the Alternating Series Test. The sum of the first 5 terms of the series is

$$s_5 = -\frac{1}{4} + \frac{1}{32} - \frac{1}{384} + \frac{1}{6144} - \frac{1}{122880} \approx -0.2211995.$$

So we know that

$$\begin{array}{rcl} & |R_5| & < |a_6| = \frac{1}{2949120} \approx 0.0000003 \\ -0.0000003 & < s - s_5 & < 0.0000003 \\ -0.2211998 & < s & < -0.2211992 \end{array}$$

where s is the sum of the series.

3. (a) We use the Ratio Test, letting

$$a_i = (-1)^i \frac{i^3}{3^i} \quad \text{so} \quad a_{i+1} = (-1)^{i+1} \frac{(i+1)^3}{3^{i+1}}.$$

Then

$$\lim_{i \rightarrow \infty} \left| \frac{a_{i+1}}{a_i} \right| = \lim_{i \rightarrow \infty} \frac{(i+1)^3}{3^{i+1}} \cdot \frac{3^i}{i^3} = \lim_{i \rightarrow \infty} \frac{i^3 + 3i^2 + 3i + 1}{3i^3} = \frac{1}{3} = L.$$

Since $L < 1$, the given series converges.

(b) We use the Ratio Test, letting

$$a_i = \frac{4^i}{i!} \quad \text{so} \quad a_{i+1} = \frac{4^{i+1}}{(i+1)!}.$$

Then

$$\lim_{i \rightarrow \infty} \left| \frac{a_{i+1}}{a_i} \right| = \lim_{i \rightarrow \infty} \frac{4^{i+1}}{(i+1)!} \cdot \frac{i!}{4^i} = \lim_{i \rightarrow \infty} \frac{4}{i+1} = 0 = L.$$

Since $L < 1$, the given series converges.

(c) We use the Root Test, letting $a_i = \left(\frac{3i}{i+2} \right)^i$. Then

$$\lim_{i \rightarrow \infty} |a_i|^{\frac{1}{i}} = \lim_{i \rightarrow \infty} \frac{3i}{i+2} = 3 = L.$$

Since $L > 1$, the given series diverges.

(d) We use the Root Test, letting $a_i = i \left(\frac{1}{7}\right)^{2i}$. Then

$$\lim_{i \rightarrow \infty} |a_i|^{\frac{1}{i}} = \lim_{i \rightarrow \infty} i^{\frac{1}{i}} \left(\frac{1}{7}\right)^2 = \frac{1}{49} \lim_{i \rightarrow \infty} i^{\frac{1}{i}} = \frac{1}{49} = L.$$

Since $L < 1$, the given series converges.

(e) We use the Ratio Test, letting

$$a_i = \frac{i^i}{i!} \quad \text{so} \quad a_{i+1} = \frac{(i+1)^{i+1}}{(i+1)!}.$$

Then

$$\lim_{i \rightarrow \infty} \left| \frac{a_{i+1}}{a_i} \right| = \lim_{n \rightarrow \infty} \frac{(i+1)^{i+1}}{(i+1)!} \cdot \frac{i!}{i^i} = \lim_{i \rightarrow \infty} \frac{(i+1)^{i+1}}{(i+1)i^i} = \lim_{i \rightarrow \infty} \frac{(i+1)^i}{i^i} = \lim_{i \rightarrow \infty} \left(\frac{i+1}{i} \right)^i.$$

This is a 1^∞ indeterminate form, so we let $f(x) = \left(\frac{x+1}{x}\right)^x$ and set

$$\begin{aligned} y &= \left(\frac{x+1}{x}\right)^x \\ \ln(y) &= x \ln\left(\frac{x+1}{x}\right) = \frac{\ln\left(\frac{x+1}{x}\right)}{\frac{1}{x}} \\ \lim_{x \rightarrow \infty} \ln(y) &= \lim_{x \rightarrow \infty} \frac{\ln\left(\frac{x+1}{x}\right)}{\frac{1}{x}} \stackrel{\text{H}}{=} \lim_{x \rightarrow \infty} \frac{\frac{x}{x+1} \left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x \rightarrow \infty} \frac{x+1}{x} = 1 \end{aligned}$$

so

$$\lim_{i \rightarrow \infty} \left| \frac{a_{i+1}}{a_i} \right| = e^1 = e = L,$$

and since $L > 1$, the series diverges.

(f) Let $a_i = \frac{1}{i^3 \sqrt{i}} = \frac{1}{i^{\frac{7}{2}}}$. First we try the Ratio Test, for which $a_{i+1} = \frac{1}{(i+1)^{\frac{7}{2}}}$:

$$\lim_{i \rightarrow \infty} \left| \frac{a_{i+1}}{a_i} \right| = \lim_{i \rightarrow \infty} \frac{1}{(i+1)^{\frac{7}{2}}} \cdot i^{\frac{7}{2}} = \lim_{i \rightarrow \infty} \left(\frac{i}{i+1} \right)^{\frac{7}{2}} = 1^{\frac{7}{2}} = 1 = L,$$

so the Ratio Test fails. So instead we try the Root Test:

$$\lim_{i \rightarrow \infty} |a_i|^{\frac{1}{i}} = \lim_{i \rightarrow \infty} \frac{1}{i^{\frac{7}{2i}}} = \lim_{i \rightarrow \infty} \left(\frac{1}{i^{\frac{1}{i}}} \right)^{\frac{7}{2}} = 1^{\frac{7}{2}} = 1 = L,$$

so the Root Test also fails. (In fact, it can easily be seen that the given series is a convergent p -series.)

(g) We use the Root Test, letting $a_i = (-1)^{i+1} \left(\frac{3}{2} - \sqrt[i]{i}\right)^i$. Then

$$\lim_{i \rightarrow \infty} |a_i|^{\frac{1}{i}} = \lim_{i \rightarrow \infty} \left(\frac{3}{2} - \sqrt[i]{i}\right) = \lim_{i \rightarrow \infty} \left(\frac{3}{2} - i^{\frac{1}{i}}\right) = \frac{3}{2} - 1 = \frac{1}{2} = L.$$

So since $L < 1$, the given series converges.

(h) We use the Ratio Test, letting

$$a_i = \frac{2^i}{i^2 + 6} \quad \text{so} \quad a_{i+1} = \frac{2^{i+1}}{(i+1)^2 + 6}.$$

Then

$$\lim_{i \rightarrow \infty} \left| \frac{a_{i+1}}{a_i} \right| = \lim_{i \rightarrow \infty} \frac{2^{i+1}}{(i+1)^2 + 6} \cdot \frac{i^2 + 6}{2^i} = \lim_{i \rightarrow \infty} \frac{2(i^2 + 6)}{(i+1)^2 + 6} = 2 = L.$$

So since $L > 1$, the given series diverges.

(i) We use the Ratio Test, letting

$$a_i = \frac{(4i)!}{(i!)^3} \quad \text{so} \quad a_{i+1} = \frac{(4i+4)!}{((i+1)!)^3}.$$

Then

$$\begin{aligned} \lim_{i \rightarrow \infty} \left| \frac{a_{i+1}}{a_i} \right| &= \lim_{i \rightarrow \infty} \frac{(4i+4)!}{((i+1)!)^3} \cdot \frac{(i!)^3}{(4i)!} \\ &= \lim_{i \rightarrow \infty} \frac{(4i+1)(4i+2)(4i+3)(4i+4)}{(i+1)^3} = \infty = L. \end{aligned}$$

Since $L > 1$, the given series diverges.