MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 1.5 **Math 2000 Worksheet** WINTER 2020

SOLUTIONS

1. (a) We use the Direct Comparison Test with the convergent p-series $\sum_{i=1}^{\infty} \frac{5}{i^3}$ (recalling that multiplying by the constant 5 doesn't affect the convergence of the series). Then for $i \geq 1$

$$\begin{aligned} i^3 &< i^3 + 2 \\ \frac{1}{i^3} &> \frac{1}{i^3+2} \\ \frac{5}{i^3} &> \frac{5}{i^3+2}, \end{aligned}$$

so the given series converges as well.

(b) We use the Direct Comparison Test with the harmonic series $\sum_{i=2}^{\infty} \frac{1}{i}$, which is divergent. We have for $i \geq 2$,

$$\begin{aligned} i &> i - \sqrt{i} \\ \frac{1}{i} &< \frac{1}{i - \sqrt{i}} \end{aligned}$$

and so the given series also diverges.

(c) We use the Limit Comparison Test with the convergent p-series $\sum_{i=1}^{\infty} \frac{1}{i^2}$:

$$\lim_{i\to\infty} \frac{\frac{1}{3i^2-1}}{\frac{1}{i^2}} = \lim_{i\to\infty} \frac{i^2}{3i^2-1} = \frac{1}{3},$$

so the given series converges as well.

(d) We use the Direct Comparison Test with the divergent geometric series $\sum_{i=1}^{\infty} \left(\frac{5}{4}\right)^i$. For $i \geq 1$,

$$\begin{aligned} 5^i &< 6 + 5^i \\ \frac{5^i}{4^i} &< \frac{6+5^i}{4^i} \\ \left(\frac{5}{4}\right)^i &< \frac{6+5^i}{4^i}. \end{aligned}$$

Hence the given series is divergent.

(e) We use the Limit Comparison Test with the convergent geometric series $\sum_{i=0}^{\infty}\left(\frac{5}{6}\right)^i = \sum_{i=0}^{\infty}\frac{5^i}{6^i}$:

$$\lim_{i\to\infty}\frac{\frac{1+5^i}{1+6^i}}{\frac{5^i}{6^i}} = \frac{\left(\frac{1}{5}\right)^i+1}{\left(\frac{1}{6}\right)^i+1} = 1,$$

so the given series is convergent.

(f) We use the Limit Comparison Test with the (divergent) harmonic series $\sum_{i=1}^{\infty}\frac{1}{i}$:

$$\lim_{i\to\infty}\frac{\frac{i+4}{(i+2)^2}}{\frac{1}{i}} = \lim_{i\to\infty}\frac{i^2+4i}{i^2+4i+4} = 1,$$

so the given series diverges.

(g) We use the Direct Comparison Test with the convergent p-series $\sum_{i=1}^{\infty}\frac{1}{i^{\frac{5}{2}}} = \sum_{i=1}^{\infty}\frac{1}{\sqrt{i^5}}$. Then for $i \geq 1$,

$$i^5+1 > i^5$$

$$\sqrt{i^5+1} > \sqrt{i^5}$$

$$\frac{1}{\sqrt{i^5+1}} < \frac{1}{\sqrt{i^5}}$$

so the given series is convergent.

(h) We use the Limit Comparison Test with the divergent geometric series $\sum_{i=1}^{\infty}2^i$:

$$\lim_{i\to\infty}\frac{\frac{2^i(4i^3-1)}{i^3+i^2+11}}{2^i} = \lim_{i\to\infty}\frac{4i^3-1}{i^3+i^2+11} = 4,$$

so the given series diverges too.

(i) First note that

$$\frac{i!}{i^i} = \frac{1\cdot 2\cdot 3\cdot 4\cdots i}{i\cdot i\cdot i\cdot i\cdots i} = \frac{1}{i}\cdot\frac{2}{i}\cdot\frac{3}{i}\cdot\frac{4}{i}\cdots\frac{k}{i}\cdots\frac{i}{i},$$

so every factor $\frac{k}{i} \leq 1$. Hence we can conclude that

$$\frac{i!}{i^i} \leq \frac{1}{i}\cdot\frac{2}{i}\cdot 1\cdot 1\cdots 1 = \frac{2}{i^2}.$$

Since $\sum_{i=1}^{\infty}\frac{2}{i^2}$ is a convergent p-series, by the Direct Comparison Test, the given series converges as well.