MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 1.4 **Math 2000 Worksheet** WINTER 2020

SOLUTIONS

1. (a) Let $f(x) = \frac{1}{\sqrt{x}}$; clearly, this is continuous and positive for $x \geq 1$. To see that it is decreasing, we have

$$f'(x) = -\frac{1}{2}x^{-\frac{3}{2}} < 0 \quad \text{for } x \geq 1.$$

Hence the given series satisfies the requirements of the Integral Test. Then we compute

$$\int_1^\infty x^{-\frac{1}{2}}\,dx = \lim_{T\to\infty}\int_1^T x^{-\frac{1}{2}}\,dx = \lim_{T\to\infty}\left[2\sqrt{x}\right]_1^T = \lim_{T\to\infty}\left[2\sqrt{T}-2\right] = \infty.$$

Hence the given series is divergent. (Note that we knew this to be the case, since it is a p-series with $p \leq 1$.)

(b) Let $f(x) = \frac{x}{e^{5x}} = xe^{-5x}$. We see immediately that this is continuous and positive for $x \geq 1$. To check that it is decreasing, note that

$$f'(x) = e^{-5x} - 5xe^{-5x} = e^{-5x}(1-5x) < 0 \quad \text{for } x \geq 1$$

since $e^{-5x} > 0$ for all x and $1 - 5x < 0$ for $x \geq 1$. Thus we can implement the Integral Test. We compute

$$\int_1^\infty xe^{-5x}\,dx = \lim_{T\to\infty}\int_1^T xe^{-5x}\,dx.$$

We use Integration by Parts, letting $u = x$ so $du = dx$, and $dv = e^{-5x}\,dx$ so $v = -\frac{1}{5}e^{-5x}$. Then the integral becomes

$$\begin{aligned}
\lim_{T\to\infty}\left(\left[-\frac{1}{5}xe^{-5x}\right]_1^T + \frac{1}{5}\int_1^T e^{-5x}\,dx\right) &= \lim_{T\to\infty}\left[-\frac{1}{5}xe^{-5x} - \frac{1}{25}e^{-5x}\right]_1^T \\
&= \lim_{T\to\infty}\left[-\frac{1}{5}Te^{-5T} - \frac{1}{25}e^{-5T} + \frac{1}{5}e^{-5} + \frac{1}{25}e^{-5}\right] \\
&= \lim_{T\to\infty}\left[-\frac{T}{5e^{5T}} - \frac{1}{25e^{5T}} + \frac{6}{25e^5}\right] = \frac{6}{25e^5},
\end{aligned}$$

where l'Hôpital's Rule is needed to evaluate the limit of the first term in the square brackets. Hence the given series also converges.

(c) Let $f(x) = \frac{\arctan(x)}{x^2+1}$. Again, it is clear that this is continuous and positive for $x \geq 1$. To check that it is decreasing, observe that

$$f'(x) = \frac{1 - 2x\arctan(x)}{(x^2+1)^2}.$$

The denominator is positive, so we need to ensure that $1 - 2x\arctan(x) < 0$ for $x \geq 1$. Both x and $\arctan(x)$ are increasing for $x \geq 1$, so the smallest value of $2x\arctan(x)$ occurs at $x = 1$, which is $2(1)\arctan(1) = \frac{\pi}{2}$ so $1 - 2x\arctan(x) \leq 1 - \frac{\pi}{2} < 0$, as required. Hence we can use the Integral Test. We have

$$\int_1^\infty \frac{\arctan(x)}{x^2+1}\,dx = \lim_{T\to\infty}\int_1^T \frac{\arctan(x)}{x^2+1}\,dx.$$

We try a u-substitution. Let $u = \arctan(x)$ so $du = \frac{dx}{x^2+1}$; for $x = 1$, $u = \frac{\pi}{4}$ and for $x = T$, $u = \arctan(T)$. The integral becomes

$$\lim_{T\to\infty}\int_{\frac{\pi}{4}}^{\arctan(T)} u\,du = \lim_{T\to\infty}\left[\frac{1}{2}u^2\right]_{\frac{\pi}{4}}^{\arctan(T)} = \lim_{T\to\infty}\left[\frac{1}{2}\arctan^2(T) - \frac{\pi^2}{32}\right] = \frac{\pi^2}{8} - \frac{\pi^2}{32} = \frac{3\pi^2}{32}.$$

Thus the given series converges.

(d) Let $f(x) = \frac{\ln(x)}{x^2} = x^{-2}\ln(x)$, which is continuous and positive for $x \geq 2$. Then

$$f'(x) = -2x^{-3}\ln(x) + x^{-3} = \frac{1 - 2\ln(x)}{x^3}.$$

Here, for $x \geq 2$ the denominator is positive and $1 - 2\ln(x) < 1 - 2\ln(2) < 0$, so $f(x)$ is decreasing for $x \geq 2$. Applying the Integral Test, we compute

$$\int_2^\infty \frac{\ln(x)}{x^2}\,dx = \lim_{T\to\infty}\int_2^T \frac{\ln(x)}{x^2}\,dx.$$

We try Integration by Parts with $u = \ln(x)$ so $du = \frac{dx}{x}$ and $dv = \frac{dx}{x^2}$ so $v = -\frac{1}{x}$. The integral becomes

$$\lim_{T\to\infty}\left(\left[-\frac{\ln(x)}{x}\right]_2^T + \int_2^T \frac{dx}{x^2}\right) = \lim_{T\to\infty}\left[-\frac{\ln(x)}{x} - \frac{1}{x}\right]_2^T$$
$$= \lim_{T\to\infty}\left[-\frac{\ln(T)}{T} - \frac{1}{T} + \frac{\ln(2)+1}{2}\right] = \frac{\ln(2)+1}{2},$$

where the limit of the first term in the square brackets can be evaluated using l'Hôpital's Rule. Hence the given series converges.

(e) Let $f(x) = \frac{\ln(x)}{x} = x^{-1}\ln(x)$. Once again, we readily see that this is continuous and positive for $x \geq 2$. Also,

$$f'(x) = -x^{-2}\ln(x) + x^{-2} = \frac{1 - \ln(x)}{x^2} < 0 \quad \text{for } x \geq 3,$$

so $f(x)$ is decreasing and we can apply the Integral Test. We have

$$\int_3^T \frac{\ln(x)}{x}\,dx = \lim_{T\to\infty}\int_3^T \frac{\ln(x)}{x}\,dx.$$

Using u-substitution, let $u = \ln(x)$ so $du = \frac{dx}{x}$; $x = 3$ gives $u = \ln(3)$ and $x = T$ gives $u = \ln(T)$. The integral becomes

$$\lim_{T\to\infty} \int_{\ln(3)}^{\ln(T)} u\,du = \lim_{T\to\infty} \left[\frac{1}{2}u^2\right]_{\ln(3)}^{\ln(T)} = \lim_{T\to\infty} \left[\frac{1}{2}\ln^2(T) - \frac{1}{2}\ln^2(3)\right] = \infty.$$

Therefore the given series is divergent.

2. Let $f(x) = \frac{1}{x^7}$. This is continuous and positive for $x \geq 1$, and so certain for $x \geq n$, regardless of what n might be. To see that it is decreasing, we have

$$f'(x) = -\frac{7}{x^8} < 0 \quad \text{for all } x \geq 1.$$

Finally, observe that the given series is a p-series with $p > 1$ and so certainly converges. Then the error after adding together the first n terms, R_n, obeys

$$R_n \leq \int_n^\infty \frac{dx}{x^7} = \lim_{T\to\infty} \int_n^T \frac{dx}{x^7} = \lim_{T\to\infty} \left[-\frac{1}{6x^6}\right]_n^T = \lim_{T\to\infty} \left[-\frac{1}{6T^6} + \frac{1}{6n^6}\right] = \frac{1}{6n^6}.$$

When $n = 2$, we have $R_n \leq 0.0026$, which provides accuracy to two decimal places. When $n = 3$, we have $R_n \leq 0.00023$, which provides accuracy to three decimal places, as desired. Then

$$s_3 = 1 + \frac{1}{2^7} + \frac{1}{3^7} \approx 1.008$$

is equal to the sum of the series, to three decimal places.