MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 1.2 | **Math 2000 Worksheet** | WINTER 2020

SOLUTIONS

1. (a) We have

$$\lim_{i\to\infty} a_i = \lim_{i\to\infty} \frac{\sqrt{i}}{2-\sqrt{i}} \cdot \frac{\frac{1}{\sqrt{i}}}{\frac{1}{\sqrt{i}}} = \lim_{i\to\infty} \frac{1}{\frac{2}{\sqrt{i}}-1} = \frac{1}{0-1} = -1,$$

so $\{a_i\}$ converges to -1.

(b) We have

$$\lim_{i\to\infty} a_i = \lim_{i\to\infty} \frac{i}{2-\sqrt{i}} \cdot \frac{\frac{1}{\sqrt{i}}}{\frac{1}{\sqrt{i}}} = \lim_{i\to\infty} \frac{\sqrt{i}}{\frac{2}{\sqrt{i}}-1} = -\infty,$$

so $\{a_i\}$ diverges.

(c) We have

$$\lim_{i\to\infty} a_i = \lim_{i\to\infty} \left[7-\left(-\frac{1}{4}\right)^i\right] = 7 - \lim_{i\to\infty}\left(-\frac{1}{4}\right)^i = 7-0=7,$$

so $\{a_i\}$ converges to 7.

(d) We have

$$a_i = \frac{3\cdot 7^i}{2^{3i-1}} = \frac{3\cdot 7^i}{2^{3i}\cdot 2^{-1}} = \frac{6\cdot 7^i}{8^i} = 6\left(\frac{7}{8}\right)^i.$$

Thus

$$\lim_{i\to\infty} a_i = 6\lim_{i\to\infty}\left(\frac{7}{8}\right)^i = 6\cdot 0 = 0,$$

so $\{a_i\}$ converges to 0.

(e) Since 5^i is the dominant term in the denominator, we can write

$$\lim_{i\to\infty} a_i = \lim_{i\to\infty} \frac{5^i+1}{5^i-1} \cdot \frac{\frac{1}{5^i}}{\frac{1}{5^i}} = \lim_{i\to\infty} \frac{1+\left(\frac{1}{5}\right)^i}{1-\left(\frac{1}{5}\right)^i} = \frac{1+0}{1-0} = 1.$$

Hence $\{a_i\}$ converges to 1.

(f) Since 3^i is the dominant term in the denominator, we can write

$$\lim_{i\to\infty} a_i = \lim_{i\to\infty} \frac{5^i+1}{3^i-2^i} \cdot \frac{\frac{1}{3^i}}{\frac{1}{3^i}} = \lim_{i\to\infty} \frac{\left(\frac{5}{3}\right)^i+\left(\frac{1}{3}\right)^i}{1-\left(\frac{2}{3}\right)^i} = \lim_{i\to\infty} \frac{\left(\frac{5}{3}\right)^i+0}{1-0} = \lim_{i\to\infty}\left(\frac{5}{3}\right)^i,$$

which does not exist because the common ratio $r=\frac{5}{3}>1$. Hence $\{a_i\}$ is divergent.

2. (a) Observe that $\sin\left(\frac{i\pi}{2}\right)$ assumes values of 1, 0, -1, 0, and then repeats. Thus the first few terms of the sequence are

$$\{2,1,0,1,2,1,0,1,\ldots\}$$

and this pattern repeats infinitely. Hence we can see that $\{a_i\}$ diverges.

(b) Note that

$$a_i = \frac{i!}{(i+2)!} = \frac{1\cdot 2\cdots i}{1\cdot 2\cdots i\cdot(i+1)\cdot(i+2)} = \frac{1}{(i+1)(i+2)} = \frac{1}{i^2+3i+2},$$

so

$$\lim_{i\to\infty} a_i = \lim_{i\to\infty}\frac{1}{i^2+3i+2} = 0.$$

Hence $\{a_i\}$ converges to 0.

(c) Recall that

$$1+2+3+\cdots+i = \frac{i(i+1)}{2}.$$

Hence

$$\frac{1}{i^2}+\frac{2}{i^2}+\cdots+\frac{i}{i^2} = \frac{1+2+\cdots+i}{i^2} = \frac{\left(\frac{i(i+1)}{2}\right)}{i^2} = \frac{i+1}{2i} = \frac{1}{2}+\frac{1}{2i}$$

so

$$\lim_{i\to\infty} a_i = \lim_{i\to\infty}\left(\frac{1}{2}+\frac{1}{2i}\right) = \frac{1}{2}$$

and $\{a_i\}$ converges to $\frac{1}{2}$.

(d) We use the Squeeze Theorem. Observe that since $0 \le \sin^2(i) \le 1$ for all i,

$$0 \le \frac{\sin^2(i)}{5^i} \le \frac{1}{5^i}.$$

But

$$\lim_{i\to\infty} 0 = 0 \quad \text{and} \quad \lim_{i\to\infty}\frac{1}{5^i} = \lim_{i\to\infty}\left(\frac{1}{5}\right)^i = 0,$$

so by the Squeeze Theorem,

$$\lim_{i\to\infty}\frac{\sin^2(i)}{5^i} = 0$$

and hence $\{a_i\}$ converges to 0.

(e) Since $\lim_{i\to\infty} a_i$ is an $\frac{\infty}{\infty}$ indeterminate form, we let $f(x) = \dfrac{\ln(2+e^x)}{9x}$ and then we use l'Hôpital's Rule:

$$\begin{aligned}\lim_{x\to\infty}\frac{\ln(2+e^x)}{9x} &\overset{\text{H}}{=} \lim_{x\to\infty}\frac{\frac{e^x}{2+e^x}}{9}\\ &= \lim_{x\to\infty}\frac{1}{\frac{18}{e^x}+9}\\ &= \frac{1}{9}.\end{aligned}$$

(Alternatively, rather than dividing through by e^x in the second-last line, we could simply apply L'Hôpital's Rule again.) Either way, we see that $\{a_i\}$ converges to $\frac{1}{9}$ by the Evaluation Theorem.

(f) Again we use the Evaluation Theorem. This time, $\lim_{i\to\infty} a_i$ is a 1^∞ indeterminate form. Thus we let $f(x) = \left(1+\frac{3}{x}\right)^x$ and write

$$y = \ln\left(1+\frac{3}{x}\right)^x = x\ln\left(1+\frac{3}{x}\right) = \frac{\ln\left(\frac{x+3}{x}\right)}{\frac{1}{x}}.$$

Now $\lim_{x\to\infty} y$ is a $\frac{0}{0}$ indeterminate form, and we can apply l'Hôpital's Rule:

$$\begin{aligned}\lim_{x\to\infty} y &\overset{\text{H}}{=} \lim_{x\to\infty} \frac{\frac{x}{x+3}\cdot\left(-\frac{3}{x^2}\right)}{-\frac{1}{x^2}} \\ &= 3\lim_{x\to\infty}\frac{x}{x+3} \\ &= 3.\end{aligned}$$

Therefore $\lim_{x\to\infty} f(x) = e^3$, and so $\{a_i\}$ converges to e^3 as well.

3. (a) To test for monotonicity, we let

$$f(x) = \frac{3x-7}{4x+1} \quad\Longrightarrow\quad f'(x) = \frac{31}{(4x+1)^2} > 0$$

for all x. Hence $\{a_i\}$ is increasing. Clearly, then, the sequence is bounded below by $a_1 = -\frac{4}{5}$. Also, $4i+1 > 3i-7$ for all $i \geq 1$, so $a_i < 1$, and therefore 1 is an upper bound. Hence $-\frac{4}{5} < a_i < 1$ and the sequence is bounded. Thus it converges by the Bounded Monotonic Sequence Theorem.

(b) The first few terms of the sequence are

$$\left\{\frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \ldots\right\}$$

and these terms cycle over and over again. Hence the sequence is not monotonic. No tail of the sequence can be monotonic either, because the oscillatory behaviour continues for all n. However, as is well known, $-1 \leq \cos\left(\frac{i\pi}{3}\right) \leq 1$ and so the sequence must be bounded.

(c) Let

$$f(x) = \frac{4\sqrt{x}}{x+5} \quad \text{so} \quad f'(x) = \frac{10-2x}{\sqrt{x}(x+5)^2}.$$

The denominator here is strictly positive, but the numerator is non-negative for $1 \leq x \leq 5$ and negative for $x > 5$. Hence we have $f'(x) \geq 0$ for $1 \leq x \leq 5$ and $f'(x) < 0$ for $x > 5$. Thus $\{a_i\}$ itself is not monotonic. However, deleting at least the first four terms results in a sequence that is decreasing, so $\{a_i\}$ does have a monotonic tail. Next, note that $x = 5$ corresponds to a maximum value of $f(x)$, namely $f(5) = \frac{2\sqrt{5}}{5} = a_5$, so $\{a_i\}$ is bounded above. Also, $a_i > 0$ for all $i \geq 1$, so since $0 < a_i < \frac{2\sqrt{5}}{5}$ the sequence is bounded. Thus it converges by the Bounded Monotonic Sequence Theorem.

(d) We have

$$a_i = \frac{1 \cdot 4 \cdot 7 \cdots (3i-2)}{3 \cdot 6 \cdot 9 \cdots (3i)} \qquad \Longrightarrow \qquad a_{i+1} = \frac{1 \cdot 4 \cdot 7 \cdots (3i-2) \cdot (3i+1)}{3 \cdot 6 \cdot 9 \cdots (3i) \cdot (3i+3)}.$$

Thus

$$\frac{a_{i+1}}{a_i} = \frac{1 \cdot 4 \cdot 7 \cdots (3i-2) \cdot (3i+1)}{3 \cdot 6 \cdot 9 \cdots (3i) \cdot (3i+3)} \cdot \frac{3 \cdot 6 \cdot 9 \cdots (3i)}{1 \cdot 4 \cdot 7 \cdots (3i-2)} = \frac{3i+1}{3i+3} < 1$$

because $3i+1 < 3i+3$ for all $i \geq 1$. Hence $\{a_i\}$ is decreasing. This means that $a_1 = \frac{1}{3}$ is an upper bound, while we note that both the numerator and denominator are positive, so 0 is a lower bound. Since $0 < a_i < \frac{1}{3}$, the sequence is bounded. Thus it converges by the Bounded Monotonic Sequence Theorem.