Section 1.10

Given a function $f(x)$, its Taylor series is

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(p)}{i!} (x-p)^i$$

and the Maclaurin series is

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} x^i .$$

eg Find the Maclaurin series for $f(x) = e^x$.

i	$f^{(i)}(x)$	$f^{(i)}(0)$	$\frac{f^{(i)}(0)}{i!} = k_i$
0	e^x	1	$\frac{1}{0!} = 1$
1	e^x	1	$\frac{1}{1!} = 1$
2	e^x	1	$\frac{1}{2!} = \frac{1}{2}$
3	e^x	1	$\frac{1}{3!} = \frac{1}{6}$

In general, we see that $k_i = \frac{1}{i!}$ so

$$e^x = \sum_{i=0}^{\infty} \frac{1}{i!} x^i$$

Its radius of convergence can be found by using

$$\rho = \lim_{i \to \infty} \left| \frac{k_{i+1}}{k_i} \right| = \lim_{i \to \infty} \frac{1}{(i+1)!} \cdot i!$$

$$= \lim_{i \to \infty} \frac{1}{i+1} = 0$$

Hence $R = \infty$ so the interval of convergence is $-\infty < x < \infty$.

eg Find the Maclaurin series for $f(x) = \sin(x)$.

i	$f^{(i)}(x)$	$f^{(i)}(0)$	$\frac{f^{(i)}(0)}{i!} = k_i$
0	$\sin(x)$	0	$\frac{0}{0!} = 0$
1	$\cos(x)$	1	$\frac{1}{1!} = 1$
2	$-\sin(x)$	0	$\frac{0}{2!} = 0$
3	$-\cos(x)$	-1	$\frac{-1}{3!} = -\frac{1}{6}$
4	$\sin(x)$	0	$\frac{0}{4!} = 0$
5	$\cos(x)$	1	$\frac{1}{5!} = \frac{1}{120}$

Thus

$$\begin{aligned}
\sin(x) &= 0 + x + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 + 0 - \frac{1}{7!}x^7 + \cdots \\
&= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots \\
&= \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!} x^{2i+1}
\end{aligned}$$

Again, $R = \infty$.

Common Maclaurin series

① $\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i$ for $|x| < 1$

② $e^x = \sum_{i=0}^{\infty} \frac{1}{i!} x^i$ for all x

③ $\sin(x) = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!} x^{2i+1}$ for all x

④ $\cos(x) = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!} x^{2i}$ for all x

⑤ $\ln(1+x) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i+1} x^{i+1}$ for $-1 < x \leq 1$

⑥ $\arctan(x) = \sum_{i=0}^{\infty} \frac{(-1)^i}{2i+1} x^{2i+1}$ for $|x| \leq 1$

eg Find the Maclaurin series for $f(x) = e^{-x^2}$.

Since $e^x = \sum_{i=0}^{\infty} \frac{1}{i!} x^i$ we have

$$e^{-x^2} = \sum_{i=0}^{\infty} \frac{1}{i!} (-x^2)^i$$

$$= \sum_{i=0}^{\infty} \frac{1}{i!} \cdot (-1)^i \cdot x^{2i} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} x^{2i}.$$

It will also converge for all x.

The first few terms of a Taylor series can be said to constitute a Taylor polynomial. The nth Taylor polynomial is given by

$$\sum_{i=0}^{n} \frac{f^{(i)}(p)}{i!} (x-p)^i$$

$$= \frac{f(p)}{0!} + \frac{f'(p)}{1!}(x-p) + \frac{f''(p)}{2!}(x-p)^2 + \ldots$$

eg The 3rd Taylor polynomial of $\sin(x)$ centred at $x=0$ is

$$T_3(x) = 0 + x + 0 - \frac{1}{3!} x^3$$

$$= x - \frac{1}{6} x^3.$$

eg For $f(x) = \ln(x)$ find the 4th Taylor polynomial centred at $x=1$. Use it to approximate $\ln(1.1)$.

i	$f^{(i)}(x)$	$f^{(i)}(1)$	$\frac{f^{(i)}(1)}{i!} = k_i$
0	$\ln(x)$	0	$\frac{0}{0!} = 0$
1	$1/x$	1	$\frac{1}{1!} = 1$
2	$-1/x^2$	-1	$\frac{-1}{2!} = -\frac{1}{2}$
3	$2/x^3$	2	$\frac{2}{3!} = \frac{1}{3}$
4	$-6/x^4$	-6	$\frac{-6}{4!} = -\frac{1}{4}$

$$T_4(x) = 0 + 1\cdot(x-1) + \left(-\tfrac{1}{2}\right)\cdot(x-1)^2 + \tfrac{1}{3}(x-1)^3 + \left(-\tfrac{1}{4}\right)\cdot(x-1)^4$$
$$= (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \tfrac{1}{4}(x-1)^4$$

Hence $\ln(1.1) \approx T_4(1.1) \approx 0.0953$.