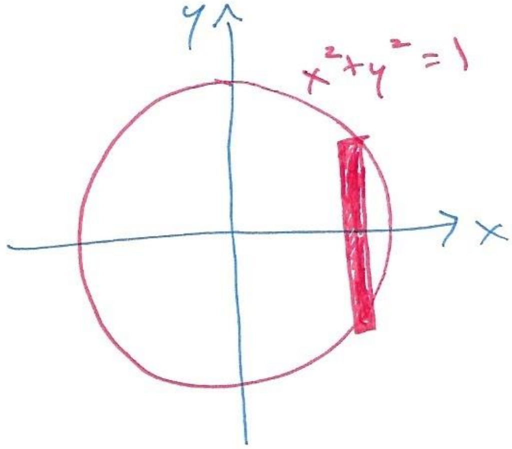
eg (cont.) Find the volume of the hemisphere $z=\sqrt{1-x^2-y^2}$.



$$V=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\,dy\,dx$$

We can view this integral as having the form $\int \sqrt{k^2-y^2}\,dy$ where $k^2=1-x^2$ so $k=\sqrt{1-x^2}$. We use a trigonometric substitution of

$$y=k\sin(\theta)=\sqrt{1-x^2}\sin(\theta)$$

$$\sqrt{1-x^2-y^2}=\sqrt{1-x^2-(1-x^2)\sin^2(\theta)}$$

$$=\sqrt{(1-x^2)(1-\sin^2(\theta))}$$

$$=\sqrt{(1-x^2)\cos^2(\theta)}=\sqrt{1-x^2}\cos(\theta)$$

$$dy=\sqrt{1-x^2}\cos(\theta)\,d\theta$$

When $y=\sqrt{1-x^2}$, $\quad\sqrt{1-x^2}=\sqrt{1-x^2}\sin(\theta)$

$$\sin(\theta)=1 \rightarrow \theta=\pi/2$$

$y=-\sqrt{1-x^2}$, $\quad -\sqrt{1-x^2}=\sqrt{1-x^2}\sin(\theta)$

$$\sin(\theta)=-1 \rightarrow \theta=-\pi/2$$

Now the integral becomes

$$V = \int_{-1}^{1} \int_{-\pi/2}^{\pi/2} \sqrt{1-x^2}\cos(\theta) \cdot \sqrt{1-x^2}\cos(\theta)\, d\theta\, dx$$

$$= \int_{-1}^{1} \int_{-\pi/2}^{\pi/2} (1-x^2)\cos^2(\theta)\, d\theta\, dx$$

$$= \int_{-1}^{1} \int_{-\pi/2}^{\pi/2} (1-x^2) \cdot \frac{1+\cos(2\theta)}{2}\, d\theta\, dx$$

$$= \int_{-1}^{1} \cancel{\int_{-\pi/2}^{\pi/2}} \frac{1-x^2}{2} \cdot \left[\theta + \tfrac{1}{2}\sin(2\theta)\right]_{\theta=-\pi/2}^{\theta=\pi/2} dx$$

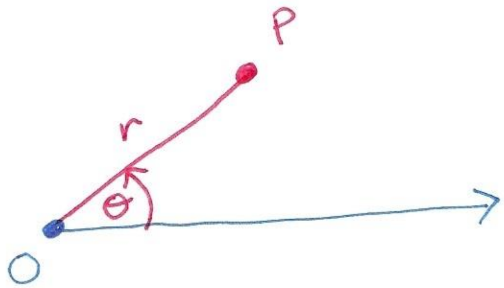
$$= \int_{-1}^{1} \frac{1-x^2}{2} \cdot \pi\, dx$$

$$= \frac{\pi}{2}\left[x - \tfrac{1}{3}x^3\right]_{-1}^{1} \boxed{= \frac{2\pi}{3}}$$

Section 2.8: Polar Coordinates

As we have seen, circular and spherical shapes are difficult to work with in the xy- or Cartesian coordinate system. For problems involving these shapes it is often useful to define an alternative called the polar coordinate system.

First, we choose a point called the origin or the pole. Typically, this is chosen to coincide with the Cartesian origin, O. Then we draw a semi-infinite line which starts at the pole, called the polar axis. We usually draw it to be the same as the positive x-axis.



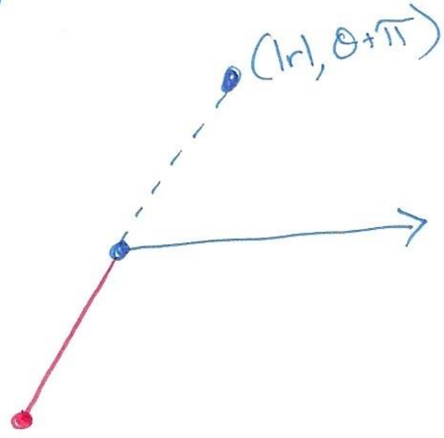
Let r be the straight-line distance from O to a point P.

Let θ be the angle (measured counterclockwise) between the line OP and the polar axis.

The pair (r, θ) are the <u>polar coordinates</u> of P.

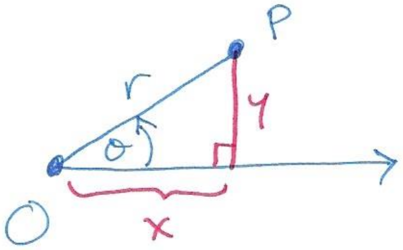
Observe that the same point P can be represented by (r, θ), by $(r, \theta + 2\pi)$, $(r, \theta + 4\pi)$, $(r, \theta - 2\pi)$, and in general by $(r, \theta + 2k\pi)$ for any integer k.

We normally assume $r > 0$ but if $r < 0$ then we define
$$(r, \theta) = (|r|, \theta + \pi)$$
that is, the point which is the same distance from O but lying directly opposite it.



$P(r, \theta)$ for $r < 0$

Observe that the pole O is described by $(0, \theta)$ for any θ.



If P has Cartesian coordinates (x,y) and polar coordinates (r,θ) then

$$r^2 = x^2 + y^2 \qquad \tan(\theta) = \frac{y}{x}$$

$$r = \sqrt{x^2+y^2} \qquad \theta = \arctan\left(\frac{y}{x}\right)$$

Likewise, $\cos(\theta) = \frac{x}{r}$ so $x = r\cos(\theta)$

$\sin(\theta) = \frac{y}{r}$ so $y = r\sin(\theta)$.

eg Suppose P has polar coordinates $(2, \pi/6)$.

$$x = 2\cos(\pi/6) = 2 \cdot \sqrt{3}/2 = \sqrt{3}$$

$$y = 2\sin(\pi/6) = 2 \cdot 1/2 = 1$$

So, in Cartesian coordinates, P is the point $(\sqrt{3}, 1)$.

Common graphs in polar coordinates:

① $r = k$ for some constant k

Its graph is a circle, centred at the origin, of radius k.

② $\theta = k$ for some constant k

Its graph is a line through the origin that makes an angle k with the polar axis.

<u>NOTE</u>: There is no page 5!