Section 1.9

We can use term-by-term differentiation and integration to generate additional power series representations.

eg $f(x) = \dfrac{x}{(1-x^2)^2}$

Observe that

$$\int \frac{x}{(1-x^2)^2}\,dx \qquad u = 1-x^2, \quad -\tfrac{1}{2}\,du = x\,dx$$

$$= -\frac{1}{2}\int \frac{1}{u^2}\,du$$

$$= -\frac{1}{2}\left[-\frac{1}{u}\right] + C$$

$$= \frac{1}{2(1-x^2)} + C \qquad \text{so} \qquad \left[\frac{1}{2(1-x^2)}\right]' = \frac{x}{(1-x^2)^2} = f(x)$$

Using $\dfrac{1}{1-x} = \sum_{i=0}^{\infty} x^i$, we can write

$$\frac{1}{2(1-x^2)} = \frac{1}{2}\cdot\frac{1}{1-x^2}$$

$$= \frac{1}{2}\sum_{i=0}^{\infty}(x^2)^i$$

$$= \frac{1}{2}\sum_{i=0}^{\infty} x^{2i}$$

$$f(x) = \left[\frac{1}{2}\sum_{i=0}^{\infty} x^{2i}\right]'$$

$$= \frac{1}{2}\sum_{i=0}^{\infty}\left[x^{2i}\right]'$$

$$= \frac{1}{2}\sum_{i=0}^{\infty} 2i\,x^{2i-1}$$

$$= \sum_{i=0}^{\infty} i\,x^{2i-1}$$

This power series must converge for

$$-1 < x^2 < 1$$

so $\quad -1 < x < 1$

However, because we have differentiated term-by-term, we must check the endpoints $x=1$ and $x=-1$.

For $x=1$, the power series becomes

$$\sum_{i=0}^{\infty} i \cdot 1^{2i-1} = \sum_{i=0}^{\infty} i$$

which diverges by the Divergence Test.

For $x=-1$, it becomes

$$\sum_{i=0}^{\infty} i \cdot (-1)^{2i-1} = -\sum_{i=0}^{\infty} i$$

which also diverges by the Divergence Test.

Hence the interval of convergence is $-1 < x < 1$.

eg $f(x) = \ln(5+x)$

Observe that $f'(x) = \frac{1}{5+x}$ so $\int \frac{1}{5+x}\,dx = f(x)$.

Now we find a power series representation for $\frac{1}{5+x}$:

$$\frac{1}{5+x} = \frac{1}{5} \cdot \frac{1}{1+x/5} = \frac{1}{5} \cdot \frac{1}{1-(-x/5)}$$

$$= \frac{1}{5} \sum_{i=0}^{\infty} \left(-\frac{x}{5}\right)^i$$

$$= \frac{1}{5} \sum_{i=0}^{\infty} \frac{(-1)^i}{5^i} x^i$$

$$= \sum_{i=0}^{\infty} \frac{(-1)^i}{5^{i+1}} x^i$$

So now $f(x) = \int \left[\sum_{i=0}^{\infty} \frac{(-1)^i}{5^{i+1}} x^i \right] dx$

$$= C + \sum_{i=0}^{\infty} \left[\frac{(-1)^i}{5^{i+1}} \cdot \int x^i dx \right]$$

$$= C + \sum_{i=0}^{\infty} \frac{(-1)^i}{5^{i+1}} \cdot \frac{x^{i+1}}{i+1}$$

$$= C + \sum_{i=0}^{\infty} \frac{(-1)^i}{5^{i+1}(i+1)} x^{i+1}$$

Observe that $f(0) = \ln(5)$ so $C = \ln(5)$. Thus the power series representation is

$$f(x) = \ln(5) + \sum_{i=0}^{\infty} \frac{(-1)^i}{5^{i+1}(i+1)} x^{i+1}.$$

It will converge for

$$-1 < -\frac{x}{5} < 1$$

$$5 > x > -5$$

At $x=5$, the power series becomes $\ln(5) + \sum_{i=0}^{\infty} \frac{(-1)^i}{5^{i+1}(i+1)} \cdot 5^{i+1}$

$$= \ln(5) + \sum_{i=0}^{\infty} \frac{(-1)^i}{i+1}$$

This is an alternating series with $p_i = \frac{1}{i+1}$. Note that $\lim_{i \to \infty} p_i = 0$ and $\{p_i\}$ is decreasing. Hence the power series converges by the Alt. Series Test.

At $x=-5$, it becomes $\ln(5) + \sum_{i=0}^{\infty} \frac{(-1)^i}{5^{i+1}(i+1)} \cdot (-5)^{i+1}$

$$= \ln(5) + \sum_{i=0}^{\infty} \frac{(-1)^i}{5^{i+1}(i+1)} \cdot (-1)^{i+1} \cdot 5^{i+1}$$

$$= \ln(5) + \sum_{i=0}^{\infty} \frac{-1}{i+1}$$

$$= \ln(5) - \sum_{i=0}^{\infty} \frac{1}{i+1}$$

This diverges by the LCT with $\sum t_i = \sum \frac{1}{i}$.

Hence the interval of convergence is $-5 < x \leq 5$.

Section 1.10: Taylor and Maclaurin Series

Given a function $f(x)$, suppose it can be represented by the power series

$$f(x) = \sum_{i=0}^{\infty} k_i (x-p)^i$$

for an appropriate choice of coefficients k_i and centre p.

Then we could also write

$$f'(x) = \sum_{i=0}^{\infty} k_i \cdot i (x-p)^{i-1}$$

$$f''(x) = \sum_{i=0}^{\infty} k_i \cdot i(i-1)(x-p)^{i-2}$$

$$f'''(x) = \sum_{i=0}^{\infty} k_i \cdot i(i-1)(i-2)(x-p)^{i-3}$$

and so on.

Therefore $f(p) = k_0$

$$f'(p) = k_1$$

$$f''(p) = 2k_2 \quad \text{so} \quad k_2 = \frac{1}{2} f''(p)$$

$$f'''(p) = 6k_3 \quad \text{so} \quad k_3 = \frac{1}{6} f'''(p)$$

In general,

$$f^{(n)}(p) = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1 \cdot k_n$$
$$= n!\, k_n$$

so $k_n = \frac{1}{n!} f^{(n)}(p)$. These are called the Taylor coefficients of the power series and the power series representation of $f(x)$ as

$$f(x) = \sum_{i=0}^{\infty} \frac{1}{i!} f^{(i)}(p)\,(x-p)^i$$

is called the Taylor series for $f(x)$, centred at $x=p$.

In the case where the centre is $x=0$, this becomes

$$f(x) = \sum_{i=0}^{\infty} \frac{1}{i!} f^{(i)}(0)\, x^i$$

which is the Maclaurin series for $f(x)$.