Section 1.9: Representing Functions as Power Series

Observe that the simplest power series is centred at $x=0$ and has $k_i = 1$ for all i. This power series has the form

$$\sum_{i=0}^{\infty} k_i (x-p)^i = \sum_{i=0}^{\infty} x^i .$$

This is a geometric series with common ratio x. It will converge when $|x| < 1$ so $-1 < x < 1$. Furthermore when this power series converges, its sum is

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x} .$$

Thus the <u>power series representation</u> of the function $f(x) = \frac{1}{1-x}$ is

$$f(x) = \sum_{i=0}^{\infty} x^i \quad \text{for } -1<x<1 .$$

We can use this result to obtain additional power series representations.

eg $f(x) = \dfrac{1}{1+x}$

This function can be obtained from $\frac{1}{1-x}$ by replacing x with $-x$. Thus its power series representation is

$$f(x) = \sum_{i=0}^{\infty} (-x)^i = \sum_{i=0}^{\infty} (-1)^i x^i$$

which is a power series centred at $x=0$ with $k_i = (-1)^i$. It will converge when

$$-1 < -x < 1 \quad \text{so} \quad -1<x<1 .$$

To obtain the power series representation of a function $f(x)$ from the representation of $\frac{1}{1-x}$, we can replace x with any constant multiple of x or x^n (for natural numbers n) or with any other polynomial expression. We could also multiply by any constant, by x, or by x^n.

eg $f(x) = \frac{4}{1-8x^3}$

We can write $f(x)$ as $4 \cdot \frac{1}{1-8x^3}$ so

$$f(x) = 4 \sum_{i=0}^{\infty} (8x^3)^i$$
$$= 4 \sum_{i=0}^{\infty} 8^i x^{3i}$$
$$= 2^2 \sum_{i=0}^{\infty} 2^{3i} x^{3i}$$
$$= \sum_{i=0}^{\infty} 2^{3i+2} x^{3i}$$

It will converge for $-1 < 8x^3 < 1$

$$-\frac{1}{8} < x^3 < \frac{1}{8}$$
$$-\frac{1}{2} < x < \frac{1}{2}$$

eg $f(x) = \frac{1}{x}$

We can replace x in $\frac{1}{1-x}$ with $-x+1$

because $\frac{1}{1-(-x+1)} = \frac{1}{1+x-1} = \frac{1}{x}$.

Thus

$$f(x) = \sum_{i=0}^{\infty} (1-x)^i$$

$$= \sum_{i=0}^{\infty} \left[-(x-1)\right]^i$$

$$= \sum_{i=0}^{\infty} (-1)^i (x-1)^i$$

which is a power series centred at $x=1$ with $k_i = (-1)^i$.

It converges for

$$-1 < 1-x < 1$$

$$-2 < -x < 0$$

$$2 > x > 0$$

eg $f(x) = \dfrac{x^2}{3-x}$

One approach is to write this as $f(x) = x^2 \cdot \dfrac{1}{1-(x-2)}$ so

$$f(x) = x^2 \sum_{i=0}^{\infty} (x-2)^i$$

However, this cannot be written in the standard form of a power series.

Alternatively, we could write

$$f(x) = x^2 \cdot \frac{1}{3} \cdot \frac{1}{1-x/3}$$

$$= \frac{1}{3}x^2 \sum_{i=0}^{\infty} \left(\frac{x}{3}\right)^i$$

$$= \frac{1}{3}x^2 \sum_{i=0}^{\infty} \frac{1}{3^i} \cdot x^i = \sum_{i=0}^{\infty} \frac{1}{3^{i+1}} x^{i+2}$$

It converges when

$$-1 < \frac{x}{3} < 1$$

$$-3 < x < 3$$

Power series can be differentiated term-by-term. Thus

$$\left[\sum_{i=0}^{\infty} k_i (x-p)^i\right]' = \sum_{i=0}^{\infty} \left[k_i (x-p)^i\right]$$
$$= \sum_{i=0}^{\infty} k_i \cdot i (x-p)^{i-1}$$

Likewise, they can be integrated term-by-term:

$$\int \left[\sum_{i=0}^{\infty} k_i (x-p)^i\right] dx = C + \sum_{i=0}^{\infty} \left[\int k_i (x-p)^i dx\right]$$
$$= C + \sum_{i=0}^{\infty} k_i \cdot \frac{1}{i+1} (x-p)^{i+1}$$

The radii of convergence of these power series are the same as the radius of convergence of the original power series. However, the interval of convergence may change because the behaviour at the endpoints may be different than for the original power series.