Section 1.6

Theorem: The Alternating Series Test

Given an alternating series which can be written in one of the forms

$$\sum (-1)^{i} p_i \quad \text{or} \quad \sum (-1)^{i+1} p_i$$

where $p_i > 0$ for all i, then if

① $\{p_i\}$ is decreasing for all i

and ② $\lim_{i \to \infty} p_i = 0$

it must be that the alternating series converges.

eg $\displaystyle\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i}$ (the alternating harmonic series)

Here $p_i = \frac{1}{i}$ so $\{\frac{1}{i}\}$ is decreasing

and $\lim_{i \to \infty} \frac{1}{i} = 0$. Hence, by the Alt. Series Test,

the series converges.

eg $\displaystyle\sum_{i=1}^{\infty} (-1)^{i} \left[\frac{\pi}{2} - \arctan(i)\right]$

$$p_i = \frac{\pi}{2} - \arctan(i)$$

Consider $f(x) = \frac{\pi}{2} - \arctan(x)$

$$f'(x) = -\frac{1}{1+x^2} < 0 \text{ for all } x \geq 1$$

Hence $\{p_i\}$ is decreasing. Also,

$$\lim_{i \to \infty} p_i = \lim_{i \to \infty} \left[\frac{\pi}{2} - \arctan(i)\right] = \frac{\pi}{2} - \frac{\pi}{2} = 0$$

So by the Alt. Series Test, the series is convergent.

eg $\displaystyle\sum_{i=1}^{\infty} \frac{(-1)^i i}{1000+5i}$

$$p_i = \frac{i}{1000+5i}$$

$$\lim_{i\to\infty} p_i = \lim_{i\to\infty} \frac{i}{1000+5i} = \frac{1}{0+5} = \frac{1}{5}$$

Hence $\displaystyle\lim_{i\to\infty} \frac{(-1)^i i}{1000+5i}$ does not exist because the terms alternately approach $\frac{1}{5}$ and $-\frac{1}{5}$. Thus, by the Divergence Test, this series diverges.

Theorem: Given an alternating series $\sum(-1)^i p_i$ or $\sum(-1)^{i+1} p_i$

if ① $\{p_i\}$ is decreasing

and ② $\displaystyle\lim_{i\to\infty} p_i = 0$

then $R_n = |S - s_n| \leq |a_{n+1}|$.

eg $\displaystyle\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i^4}$

$$p_i = \frac{1}{i^4}$$

$$\lim_{i\to\infty} p_i = \lim_{i\to\infty} \frac{1}{i^4} = 0$$

$\{p_i\}$ is decreasing

We will use the first 4 terms of the series to estimate the sum and we will determine the accuracy of the approximation.

$$s_4 = 1 - \frac{1}{16} + \frac{1}{81} - \frac{1}{256} \approx 0.9459$$

By the theorem, $R_4 \leq |a_5| = \frac{1}{625} = 0.0016$.

Hence $0.9459 - 0.0016 \leq S \leq 0.9459 + 0.0016$

$$0.9443 \leq S \leq 0.9475$$

Def'n: Given a series $\sum a_i$, its absolute series is the series

$$\sum |a_i| = |a_1| + |a_2| + |a_3| + \cdots$$

eg The alternating harmonic series is

$$\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i}$$

Its absolute series is $\sum_{i=1}^{\infty} \left| \frac{(-1)^{i+1}}{i} \right| = \sum_{i=1}^{\infty} \frac{1}{i}$, which is the harmonic series.

Def'n: A series $\sum a_i$ is absolutely convergent if $\sum |a_i|$ converges.

eg $\sum_{i=1}^{\infty} \frac{(-1)^i}{i^2}$

Its absolute series is $\sum_{i=1}^{\infty} \left| \frac{(-1)^i}{i^2} \right| = \sum_{i=1}^{\infty} \frac{1}{i^2}$ which is a convergent p-series. Therefore the given series is absolutely convergent.

Theorem: The Absolute Series Test

For any series $\sum a_i$, if $\sum |a_i|$ converges then $\sum a_i$ also converges. In other words, absolute convergence implies convergence.

Proof: Given a series $\sum a_i$, we define

$$b_i = a_i + |a_i| = \begin{cases} 2a_i & \text{if } a_i \geq 0 \\ 0 & \text{if } a_i < 0 \end{cases}$$

Thus $0 \leq b_i \leq 2a_i = 2|a_i|$. Hence $\sum b_i$ is a positive series. Next we assume that $\sum |a_i|$ converges, and therefore $\sum 2|a_i|$ converges.

Now we apply the DCT with $\sum t_i = \sum 2|a_i|$, which converges. Since $b_i \leq t_i$, $\sum b_i$ converges.

Finally we can write

$$a_i = b_i - |a_i|$$

Since $\sum a_i = \sum b_i - \sum |a_i|$ and both series on the right are convergent, $\sum a_i$ must also converge.

This means that for any series $\sum a_i$, there are 3 possibilities:

① $\sum |a_i|$ converges, so $\sum a_i$ is absolutely convergent and hence $\sum a_i$ converges by the Abs Series Test

② $\sum |a_i|$ diverges but $\sum a_i$ converges; this means that we say $\sum a_i$ is <u>conditionally convergent</u>

③ $\sum |a_i|$ diverges and $\sum a_i$ diverges