Section 2.2

eg Determine any points at which $f(x,y) = \frac{x^2 - y^2}{2x - y}$ is discontinuous.

This will happen when $2x - y = 0$, that is, along the line $y = 2x$.

Section 2.3: Partial Derivatives

Recall that for a function $y = f(x)$ of a single variable, its derivative is defined to be

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Now consider a function $z = f(x,y)$. We could consider the rate of change of z with respect to x, given by

$$\frac{\partial}{\partial x}[f(x,y)] = \frac{\partial z}{\partial x} = f_x(x,y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

which is the partial derivative with respect to x.

Likewise, we could consider the rate of change of z with respect to y, given by

$$\frac{\partial}{\partial y}[f(x,y)] = \frac{\partial z}{\partial y} = f_y(x,y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

which is the partial derivative with respect to y.

To compute a partial derivative with respect to x, we use our known differentiation results for a function of a single variable, and treat y as a constant. Likewise, to take a partial derivative with respect to y, we treat x as a constant.

eg $f(x,y) = x^2 y$

$f_x(x,y) = 2xy \qquad F_y(x,y) = x^2 \cdot 1 = x^2$

eg $z = \sin(x)\cos(y)\tan(x)$

$$\frac{\partial z}{\partial x} = \cos(y)\left[\frac{\partial}{\partial x}[\sin(x)]\cdot\tan(x) + \sin(x)\cdot\frac{\partial}{\partial x}[\tan(x)]\right]$$

$$= \cos(y)\left[\cos(x)\tan(x) + \sin(x)\sec^2(x)\right]$$

$$= \sin(x)\cos(y) + \sin(x)\sec^2(x)\cos(y)$$

$$\frac{\partial z}{\partial y} = \sin(x)\tan(x)\cdot\frac{\partial}{\partial y}[\cos(y)]$$

$$= -\sin(x)\tan(x)\sin(y)$$

eg $w = e^{3x}\ln(y)\sinh(z)$

$w_x = 3e^{3x}\ln(y)\sinh(z)$

$w_y = \dfrac{e^{3x}\sinh(z)}{y}$

$w_z = e^{3x}\ln(y)\cosh(z)$

Now consider a function $z = f(x,y)$ with partial derivatives z_x and z_y. Then we could obtain four <u>second-order</u> partial derivatives z_{xx}, z_{xy}, z_{yx} and z_{yy}. The derivatives z_{xy} and z_{yx} are <u>mixed partial derivatives</u>.

eg $f(x,y) = x^2 y$

$f_{xx} = 2y \qquad f_{xy} = 2x$

$f_{yy} = 0 \qquad f_{yx} = 2x$

Clairault's Theorem

If $f(x,y)$ is defined in an open circle D of a point (p,q), and the functions $f_{xy}(x,y)$ and $f_{yx}(x,y)$ are continuous in D then

$$f_{xy}(p,q) = f_{yx}(p,q)$$

eg Given $f(x,y) = \sin(x)\ln(y)$ show that

$$f_{xyy}(x,y) = f_{yyx}(x,y) = f_{yxy}(x,y)$$

$$f_x(x,y) = \cos(x)\ln(y)$$

$$f_{xy}(x,y) = \frac{\cos(x)}{y}$$

$$\boxed{f_{xyy}(x,y) = -\frac{\cos(x)}{y^2}}$$

$$f_y(x,y) = \frac{\sin(x)}{y}$$

$$f_{yy}(x,y) = -\frac{\sin(x)}{y^2} \qquad f_{yx}(x,y) = \frac{\cos(x)}{y}$$

$$\boxed{f_{yyx}(x,y) = -\frac{\cos(x)}{y^2}} \qquad \boxed{f_{yxy}(x,y) = -\frac{\cos(x)}{y^2}}$$

In Leibniz notation, we would write second-order partial derivatives as follows:

$$z_{xx} = \frac{\partial^2 z}{\partial x^2} \qquad z_{yy} = \frac{\partial^2 z}{\partial y^2}$$

$$z_{xy} = z_{yx} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

Likewise, given $z = f(x,y)$, we would write

$$f_{xyy}(x,y) = \frac{\partial^3}{\partial x \partial y^2}\left[f(x,y)\right]$$